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Edited by
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HISTORY

Tropfke, Johannes. **Geschichte der Elementar-Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter. Band 4, Ebene Geometrie.** Third Edition. Walter de Gruyter and Co., Berlin, 1940. 316 pp.

This volume deals with plane geometry. The fourth edition [published by K. Vogel] is more than twice the size of the third. *O. Neugebauer* (Providence, R. I.).

Miller, G. A. A ninth lesson in the history of mathematics. Nat. Math. Mag. 19, 64–72 (1944). [MF 11484]

Neugebauer, O. The history of ancient astronomy: problems and methods. Journal of Near Eastern Studies 4, 1–38 (1945). [MF 11838]

Expository article.

Chatley, Herbert. Ancient Egyptian star tables and the dekans. Observatory 65, 121–125 (1943). [MF 11388]

Expository article.

O. Neugebauer (Providence, R. I.).

Sachs, A. J. Some metrological problems in Old-Babylonian mathematical texts. Bull. Amer. Schools of Oriental Research no. 96, 29–39 (1944). [MF 11527]

Relations between different units of measurement play a great role in Old-Babylonian mathematical texts; their tacit assumption is frequently a major stumbling block for the modern interpretation. The author succeeds in eliminating one cause of trouble in examples dealing with the measurement of capacities and in finding new relations which were used in determining the capacity of a cargo boat, and for the measurement of piles of bricks.

O. Neugebauer (Providence, R. I.).

Sen Gupta, Prabodh Chandra. Hindu astronomy. Science and Culture 9, 522–526 (1944). [MF 10963]

Summary of studies in Hindu mathematics and astronomy. *O. Neugebauer* (Providence, R. I.).

Warrain, Francis.* **Essai sur l'Harmonices Mundi ou Musique du Monde de Johann Kepler. I. Fondements mathématiques de l'harmonie. Actual. Sci. Ind., no. 912. Hermann & Cie., Paris, 1942. 141 pp. (8 plates)

Warrain, Francis.* **Essai sur l'Harmonices Mundi ou Musique du Monde de Johann Kepler. II. L'harmonie planétaire d'après Kepler adaptée à nos connaissances actuelles. Actual. Sci. Ind., no. 913. Hermann & Cie., Paris, 1942. 144 pp.

This publication is not so much an historical study as an attempt to reconcile Kepler's ideas about the harmony of the universe with modern ideas. Kepler's work is considered as the foundation for a future esthetics of science.

O. Neugebauer (Providence, R. I.).

Bell, E. T. The golden and platinum proportions. Nat. Math. Mag. 19, 21–26 (1944). [MF 11152]

Loria, Gino. Perfectionnements, évolution, métamorphoses du concept de "coordonnées". Contribution à l'histoire de la géométrie analytique. Mathematica, Timişoara 20, 1–22 (1944). [MF 11158]

This is the second chapter of the paper begun in Mathematica, Timişoara 18, 125–145 (1942); these Rev. 4, 65.

Charadze, A. K. On a modification of the method of Hudde and the formula of Cardano. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 195–199 (1943). (Russian. Georgian summary) [MF 11697]

Concerning third degree equations. Hudde lived 1628–1704.

Karpinski, Louis C. Algebraic works to 1700. Scripta Math. 10, 149–169 (1944). [MF 11939]

Sarton, George. Lagrange's personality (1736–1813). Proc. Amer. Philos. Soc. 88, 457–496 (1944). [MF 11528]

Whitmore, Charles E. Mill and mathematics: an historical note. J. Hist. Ideas 6, 109–112 (1945). [MF 11770]

MacDuffee, C. C. Algebra's debt to Hamilton. Scripta Math. 10, 25–35 (1 plate) (1944). [MF 11935]

Synge, J. L. The life and early work of Sir William Rowan Hamilton. Scripta Math. 10, 13–24 (1944). [MF 11934]

Bateman, H. Hamilton's work in dynamics and its influence on modern thought. Scripta Math. 10, 51–63 (1944). [MF 11937]

Higgins, T. J. Biographies and collected works of mathematicians. Amer. Math. Monthly 51, 433–445 (1944). [MF 11244]

Rey Pastor, J. Professor George D. Birkhoff and his influence in Argentina. Revista Union Mat. Argentina 10, 65–68 (1945). (Spanish) [MF 12275]

García, Godofredo. Obituary: George D. Birkhoff. Revista Ci., Lima 46, 675–677 (1944) = Actas Acad. Ci. Lima 7, 435–437 (1944). (Spanish) [MF 11970]

Terracini, Alejandro. Obituary: Guido Fubini. 1879–1943. Revista Union Mat. Argentina 10, 27–30 (1944). (Spanish) [MF 11030]

Moore, Charles N. Obituary: Harris Hancock, in memoriam. Bull. Amer. Math. Soc. 50, 812–815 (1944). [MF 11486]

Weyl, Hermann. David Hilbert and his mathematical work. Bull. Amer. Math. Soc. 50, 612–654 (1944). [MF 11033]

Burkill, J. C. Henri Lebesgue. J. London Math. Soc. 19, 56–64 (1944). [MF 11322]

Simons, Lao Genevra. Obituary: David Eugene Smith—in memoriam. Bull. Amer. Math. Soc. 51, 40–50 (1945). [MF 11767]

Brasch, Frederick E. Obituary: David Eugene Smith. Science (N.S.) 100, 257–259 (1944). [MF 11037].

ALGEBRA

***Dörrie, Heinrich. Determinanten.** J. W. Edwards, Ann Arbor, Michigan, 1944. 216 pp. $4.40.

[Photographic reprint of the original published by Oldenbourg, München, 1940.] The first half of this book consists of an elementary exposition of the theory of determinants. The treatment is not modern. The last half is a collection of problems from elementary algebra and analytic geometry solved by means of determinants. The applications include the solution of cubic and quartic equations, rationalization of denominators, resultants, discriminants, Jacobians, linear dependence, quadratic forms, area of a triangle, volume of a tetrahedron, law of cosines, circle of Monge, problem of Steiner and some theorems on quadric surfaces. There are no problems to be solved by the reader.

C. C. MacDuffee (Madison, Wis.).

Murnaghan, F. D. An elementary presentation of the theory of quaternions. Scripta Math. 10, 37–49 (1944). [MF 11936]

The complex number $x+yi$ may be realized as a two-rowed square matrix

$$xE_2+yI=\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

and then de Moivre's theorem assumes the form

$$e^{\theta I}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The author shows how quaternions arise naturally in an attempt to generalize these results. If A is a real alternating matrix of order n, the matrix $O=e^A$ is a rotation matrix, that is, an orthogonal matrix of determinant $+1$. Since O satisfies an equation of degree n, the infinite series e^A may be transformed into a polynomial of degree less than n. When $n=4$, instead of calculating this polynomial directly, the author proceeds as follows. If

$$A=\begin{pmatrix} 0 & -r & q & l \\ r & 0 & -p & m \\ -q & p & 0 & n \\ -l & -m & -n & 0 \end{pmatrix},$$

A satisfies an equation of degree not greater than 2 if and only if (1) $(p, q, r)=\epsilon(l, m, n)$, $\epsilon=\pm1$. A matrix

$$A_1=\begin{pmatrix} 0 & -r & q & p \\ r & 0 & -p & q \\ -q & p & 0 & r \\ -p & -q & -r & 0 \end{pmatrix}=pI_1+qJ_1+rK_1$$

for which (1) is true with $\epsilon=1$ is said to be of the first class and a matrix $A_2=pI_2+qJ_2+rK_2$ for which (1) is true with $\epsilon=-1$ is said to be of the second class. If O_1 is the rotation matrix e^{A_1}, $O_1=\cos\theta E_4+\theta^{-1}\sin\theta A_1$, where $\theta^2=p^2+q^2+r^2$, and, if (α, β, γ) is the unit vector along (p, q, r),

$$O_1=\exp(\theta(\alpha I_1+\beta J_1+\gamma K_1))$$
$$=\cos\theta E_4+\sin\theta(\alpha I_1+\beta J_1+\gamma K_1).$$

Since $(\alpha I_1+\beta J_1+\gamma K_1)^2=-E_4$, the analogy between de Moivre's theorem and this last equation is apparent. Furthermore, the quaternion $xi+yj+zk+t$ has the two natural realizations (i) $xI_1+yJ_1+zK_1+tE_4$ and (ii) $xI_2+yJ_2+zK_2+tE_4$ as four-rowed matrices.

By showing that each A_1 commutes with each A_2 and that every alternating matrix A can be written in the form A_1+A_2, the author proves that every four by four rotation matrix $O=e^{A_1+A_2}=e^{A_1}e^{A_2}=O_1O_2=O_2O_1$, so that the rotation group of four by four matrices is the product of the two subgroups of rotation matrices of the first class and of the second class.

J. Williamson (Flushing, N. Y.).

Abstract Algebra

***Menger, Karl. Algebra of Analysis.** Notre Dame Mathematical Lectures, no. 3. University of Notre Dame, Notre Dame, Ind., 1944. 50 pp. $1.00.

This lithoprinted booklet gives an abstract treatment of the part of analysis which involves limit concepts only superficially and can therefore be treated algebraically. The first chapter describes an abstract algebra of functions of one variable (although variables do not occur explicitly). There are three undefined operations of addition, multiplication, and substitution, symbolized by $f+g$, $f\cdot g$, and fg. The latter denotes the result of substituting the function g in the function f. This substitution operation is associative but not commutative and obeys the two one-sided distributive laws $(f+g)h=fh+gh$ and $(f\cdot g)h=fh\cdot gh$. The existence of an identity element of substitution j such that $jf=fj=f$ is assumed, corresponding to the function $f(x)\equiv x$. With respect to the operations of addition and multiplication, the system is by assumption a commutative ring with unit element. The theory of constant functions c such that $cf=c$ for all f is developed. They form a subring closed with respect to substitution. Special functions with the formal properties of the reciprocal, negative, and exponential functions are introduced by postulate. The symbol rec 0 for the reciprocal of zero is undefined. Additional special functions with the formal properties of the logarithm, absolute value, signum, and power functions are studied after making some assumptions concerning positive and negative constants. Finally abstract trigonometric functions are introduced.

The second chapter, on the algebra of derivatives and integrals, studies the differentiation operator D and its right inverse S abstractly. These operators, which are not functions, obey the laws: (I) $D(f+g)=Df+Dg$; (II) $D(f\cdot g)=f\cdot Dg+g\cdot Df$; (III) $D(fg)=Dfg\cdot Dg$; (IV) $D(Sf)=f$. It is not assumed that $S(Df)=f$, but merely that $S(Df)\sim f$, where $\sim$ is an equivalence relation. The usual formulas for derivatives and integrals of power, exponential, logarithmic, and trigonometric functions are derived, as well as rules for integration by parts and by substitution. The third and last chapter on functions of higher rank lays the groundwork for a theory of partial derivatives and of functions of

several variables. The booklet is written in an informal style which makes for easy reading. There are several misprints. *O. Frink* (Xenia, Ohio).

Menger, Karl. Tri-operational algebra. Rep. Math. Colloquium (2) **5–6**, 3–10 (1944). [MF 11355]

After a brief exposition of the abstract system called tri-operational algebra [see the preceding review] the author derives some new results concerning instances of such algebras. Corresponding to the concrete polynomial algebras whose elements are polynomials in a commutative ring, with the substitution operation having its usual interpretation, the author shows that any tri-operational algebra A contains subalgebras whose elements are abstract "polynomials" of the form $c_0+c_1\cdot j+\cdots+c_n\cdot j^n$, where the coefficients c_k are "constant functions" of A, and j is the identity of substitution, with the formal properties of the function $f(x)\equiv x$.

The author shows that, if the ring of all constants of A is a field of characteristic 0, then abstract polynomials of A which are formally different are actually distinct. If the constants form a field of characteristic k, however, this may not be the case. If not, there is a unique simplest abstract polynomial, called the fundamental polynomial, which is equal to 0 in A. A necessary and sufficient condition is found that a given polynomial in a given field R_k be fundamental for some tri-operational algebra A having R_k for its ring of constants. The author investigates fundamental polynomials for the rings of integers reduced modulo 2 and modulo 4. He also considers extensions of tri-operational algebras. *O. Frink* (Xenia, Ohio).

Brown, Ferdinand L. Remarks concerning tri-operational algebra. Rep. Math. Colloquium (2) **5–6**, 11–15 (1944). [MF 11356]

The author investigates the simplest concrete instances of the formal system introduced by Menger and called tri-operational algebra [see the preceding review]. He shows that no tri-operational algebra containing just the three elements 0, 1 and j can satisfy Menger's postulates. If one postulate is omitted, however, the others may be satisfied with just three elements. There is just one type of algebra with four elements satisfying all the postulates, and one other type with four elements which violates just one postulate. Various independence examples are constructed. In particular, it is shown that the postulate $10=1$ is independent of the others. *O. Frink* (Xenia, Ohio).

Vaidyanathaswamy, R. Partially ordered sets. Math. Student **12**, 1–6 (1944). [MF 11846]

This is the chairman's opening speech at a symposium on partially ordered sets; cf. the three following reviews. *G. Birkhoff* (Cambridge, Mass.).

Krishnan, V. S. Partially ordered sets and projective geometry. Math. Student **12**, 7–14 (1944). [MF 11847]

The author gives an original and clear historical exposition of the discovery of the relation between projective geometries and fields, and of the relation between projective geometries and lattices. *G. Birkhoff.*

Chandrasekharan, K. Partially ordered sets and symbolic logic. Math. Student **12**, 14–24 (1944). [MF 11848]

The author expounds the lattice-theoretic interpretation of various algebras of propositions [cf. also O. Frink, Amer. Math. Monthly **45**, 210–219 (1938)]. He corrects a misinterpretation of C. I. Lewis's "strict implication" made by the reviewer [Lattice Theory, Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1940, p. 128; these Rev. **1**, 325]. *G. Birkhoff* (Cambridge, Mass.).

Pankajam, S. Group operation in certain distributive lattices. Math. Student **12**, 25–29 (1944). [MF 11849]

A method is given of defining the group operation of cyclic permutation of the elements of a finite chain in terms of the order relation on the chain. *G. Birkhoff.*

Dilworth, R. P. Dependence relations in a semi-modular lattice. Duke Math. J. **11**, 575–587 (1944). [MF11091]

MacLane's definition of dependence is generalized to apply to sets of elements of fixed rank, as well as to sets of points, of a semi-modular lattice L. Elements of L of rank at least n, together with the zero element z, form a quasi-modular lattice, that is, a lattice in which the chain law holds, as well as the law: if a covers $a\cap b\neq z$, then $a\cup b$ covers b. The author calls a set of points in such a lattice independent if for every subset T the rank of the lattice union of all elements of T is not less than the number of elements of T. A point p is said to be dependent on a set of points S if either p is in S or S has an independent subset T such that $T+p$ is dependent. An extensive theory of this dependence relation is developed, in which the notion of a normal set of points plays an important role. It is shown that the three fundamental properties of a dependence relation hold. These results are used to prove the following two theorems. (I) Every modular lattice of length three or less can be imbedded isometrically in a complemented modular lattice. (II) The lattice of closed subsets of the set of elements of rank two of the Boolean algebra of all subsets of a finite set S is isomorphic to the partition lattice of S. *O. Frink* (Xenia, Ohio).

Lorenzen, Paul. Eine Bemerkung zum Schreierschen Verfeinerungssatz. Math. Z. **49**, 647–653 (1944). [MF 11982]

The author considers a partially ordered set R, containing with any two elements a and b their g.l.b. $a\cap b$. The product ab of any two elements a and b in R is defined. This multiplication is unique, associative and satisfies the rule: if $a\leqq a'$, $b\leqq b'$, then $ab\leqq a'b'$. Finally R is to contain a subset S meeting the following requirements: if a and b are in S, then ab and $a\cap b$ are in S; if a, b, c are in S, then $a\leqq ab$ and $b\leqq ab$, and $a\leqq c$, $b\leqq c$ implies $ab\leqq c$; if a is in S, if b, c are in R, and if $c\leqq a$, then $a\cap bc=(a\cap b)c$, $a\cap cb=c(a\cap b)$. Examples for such systems may be found as follows: (1) R the set of subsets of a group G, S the set of subgroups of G; (2) R the set of binary relations over a set M, S the set of equivalence relations over M [see O. Ore, Duke Math. J. **9**, 573–627 (1942); these Rev. **4**, 128]; (3) certain derivations from multi-groups similar to (1). If $a\leqq b$ and $a'\leqq b'$ are in S, and if $b=b'a$, $a'=b'\cap a$, then the pair (b', a') is an "isomorphic part" of the pair (b, a). If a and b are in S and $a\leqq b$, and if $c\leqq b$ implies $ac=ca$, then a is normal in b; if a, b, c, a', b', c' are elements in S such that $c\leqq b\leqq a$, c is normal in a, (a', c') is an isomorphic part of (a, c), $c'\leqq b'\leqq a'$, $b=b'c$, $b'=b\cap a'$, b' is normal in a', then b is almost normal in a. Using these concepts the author succeeds in establishing a generalization of Zassenhaus's formula and of Schreier's refinement theorem for chains. *R. Baer* (Urbana, Ill.).

Dieudonné, Jean. Sur le socle d'un anneau et les anneaux simples infinis. Bull. Soc. Math. France **70**, 46–75 (1942). [MF 11889]

This paper is a unification and extension of results obtained earlier [J. Reine Angew. Math. **184**, 178–192 (1942); these Rev. **5**, 32; C. R. Acad. Sci. Paris **216**, 94–97 (1940); these Rev. **3**, 205]. The first part of the paper investigates the structure of a class of rings which have at least one minimal left ideal. The socle (lower cover, Sockel) of such a ring is defined as the union of all minimal left ideals, and it is shown that the socle is a direct sum of two-sided ideals, each of which is the direct sum of isomorphic minimal left ideals. A ring is defined to be quasi-simple if it is the direct sum of any number of minimal left ideals, all isomorphic to each other. It is shown that a simple ring having at least one minimal left ideal is quasi-simple, but not conversely. Let A, with $A^2\neq 0$, be quasi-simple, and let K be the field of endomorphisms of the minimal left ideals of A. Then it is proved that A is isomorphic to the transpose of a subring F of the ring of endomorphisms of a vector space E over K, every base of E having the power of the set of left ideals of which A is the direct sum. The subring F is completely characterized in terms of relations in a certain subspace E' of the dual of E. A necessary and sufficient condition on E' is given in order that A be simple. The structure of simple rings with minimal condition is deduced as a special case.

The second part of the paper is concerned with the introduction of a topology in F and the statement of algebraic properties of F in topological language. In a paper not accessible to the reviewer [Ann. Sci. École Norm. Sup. (3) **59**, 107–139 (1942)] the author has stated conditions on E' which define a "weak topology" in E; these are satisfied if A is simple. This weak topology induces a topology T in F. Among the results obtained are: "every left ideal of F is closed under T"; "the closure of any right ideal R of F is the right annihilator of the left annihilator of R." This last is a generalization of a theorem of M. Hall [Ann. of Math. (2) **40**, 360–369 (1939)].

S. A. Jennings (Vancouver, B. C.).

Jacobson, N. The equation $x'\equiv xd-dx=b$. Bull. Amer. Math. Soc. **50**, 902–905 (1944). [MF 11561]

Let d be a fixed element of an associative algebra $\mathfrak{A}$ over a field Φ. The mapping $x\rightarrow x'\equiv xd-dx$ is a derivation in $\mathfrak{A}$ whose constants are the elements of $\mathfrak{A}$ that commute with d. Assuming that d satisfies a polynomial equation with coefficients in Φ, the author obtains a necessary condition that b be d-integrable. If the minimum polynomial $\mu(\lambda)$ of d is relatively prime to its derivative $\mu'(\lambda)$, the condition is also sufficient and leads to an explicit formula for a solution of the equation $x'=b$. If $\mathfrak{A}$ is a simple algebra satisfying the descending chain condition for left ideals, the condition is sufficient when $\mu(\lambda)$ is a product of distinct irreducible factors in $\Phi[\lambda]$ and in certain other cases.

C. C. MacDuffee (Madison, Wis.).

Schilling, O. F. G. On a special class of Abelian functions. Bull. Amer. Math. Soc. **51**, 133–136 (1945). [MF 11829]

Let k be a totally real field of algebraic numbers of finite degree n and let $\mathfrak{o}$ be an order of maximal rank in k. Let μ be a totally negative number in k, and let $\mathfrak{O}$ be an order of maximal rank in $k(\mu^{\frac{1}{2}})$. It is shown how one can construct fields of Abelian functions of genus n admitting either $\mathfrak{o}$ or $\mathfrak{O}$ as their rings of complex multiplications. The construction depends on the use of certain Riemann matrices which were defined by Blumenthal. The proofs given are rather sketchy; thus, it is not immediately clear why (in the notation of the author) $\mathfrak{A}(\Omega_2)$ could not be the direct sum of two fields isomorphic with k, or why the condition that the τ_{ji}'s be transcendental implies that $\mathfrak{A}(\Omega_2)=k$.

C. Chevalley (Princeton, N. J.).

de Groot, J. Bemerkung über die analytische Fortsetzung in bewerteten Körpern. Nederl. Akad. Wetensch., Proc. **45**, 347–349 (1942). [MF 10400]

In this note a new proof is given of the fact that it is impossible to obtain an analytic extension of a power series with coefficients in a p-adic field. The exact result is that if $f(x)$ is a power series in x and x_0 is a point in the region of convergence then the series $\sum f^{(n)}(x_0)(x-x_0)^n/n!$ has the same region of convergence as the original series.

N. Jacobson (Baltimore, Md.).

de Groot, J., und Loonstra, F. Topologische Eigenschaften bewerteter Körper. Nederl. Akad. Wetensch., Proc. **45**, 658–664 (1942). [MF 10428]

An element x of an extension of a non-Archimedean valued field is called completely transcendental if it is not the limit of a sequence of algebraic elements. As is well known, any field can be obtained from its prime field by a succession of simple transcendental extensions followed by a succession of algebraic extensions. The authors show that a field with a nontrivial non-Archimedean valuation is separable if and only if it is possible to choose the (transfinite) sequence of transcendental extensions in such a way that there is only a denumerable number of elements which are completely transcendental over the field that they extend. Using a result of Zippin [Amer. J. Math. **57**, 327–341 (1935)] it is proved that the space of such a field can be made compact by adjoining a denumerable number of points.

N. Jacobson (Baltimore, Md.).

Loonstra, F. Im Kleinen kompakte nichtarchimedisch bewertete Körper. Nederl. Akad. Wetensch., Proc. **45**, 665–668 (1942). [MF 10429]

This is an expository article dealing mainly with results that were obtained originally by van Dantzig [Studien over Topologische Algebra (Thesis, University of Groningen), Amsterdam, 1931].

N. Jacobson (Baltimore, Md.).

Loonstra, F. Pseudokonvergente Folgen in nichtarchimedisch bewerteten Körpern. Nederl. Akad. Wetensch., Proc. **45**, 913–917 (1942). [MF 10441]

A sequence $\{a_i\}$ of elements of a non-Archimedean valued field K is called pseudo-convergent if for all sufficiently large i either $|a_{i+1}-a_i|<|a_i-a_{i-1}|$ or $a_i=a_{i+1}$. If $\{a_i\}$ is pseudo-convergent then $\lim |a_i|$ exists. Pseudo-convergent sequences have been used by Ostrowski [Math. Z. **39**, 269–404 (1934), in particular, pp. 368–374] to obtain valuations of a simple transcendental extension $K(x)$ of K. The present paper gives a new proof of the main theorem used by Ostrowski for this purpose, namely, if $\{a_i\}$ is pseudo-convergent and $f(x)$ is a polynomial then $\{f(a_i)\}$ is pseudo-convergent.

N. Jacobson (Baltimore, Md.).

Wade, L. I. Two types of function field transcendental numbers. Duke Math. J. **11**, 755–758 (1944). [MF 11577]

Continuation of ideas and problems developed in two earlier papers [same J. **8**, 701–720 (1941); **10**, 587–594

(1943); these Rev. 3, 263; 5, 89]. All three papers utilize certain expressions introduced into the theory by Carlitz. For methods and terms, such as the use of "transcendental" in this connection, compare the former papers and references. "Let p denote a fixed prime, $GF(p^n)$ the finite field of order p^n, $n=1, 2, \cdots$, and Γ the algebraic closure of $GF(p)$. For e indeterminates $x_1, \cdots, x_e$, $\Gamma(x_1, \cdots, x_e)$ will denote the field of rational functions in $x_1, \cdots, x_e$ with coefficients from Γ. If $E\neq 0$ and $G\neq 0$ are two polynomials in $x_1, \cdots, x_e$ (with coefficients from Γ, or, what is the same thing, from some $GF(p^n)$), define $\deg E/G = \deg E - \deg G$, where deg is an abbreviation for degree. If we write $-\deg 0=\infty$, then $-\deg$ defines an exponential valuation of $\Gamma(x_1, \cdots, x_e)$. Denote by Φ the corresponding completion of $\Gamma(x_1, \cdots, x_e)$. Here we shall consider the transcendence over $\Gamma(x_1, \cdots, x_e)$, or equivalence over $GF(p; x_1, \cdots, x_e)$ of two types of elements of Φ defined by infinite series." (I) The real numbers $\sum g^{-q^k}$, $k=0, 1, 2, \cdots$; $g>1$, $q>1$ rational integers, are known to be transcendental. For the corresponding series in Φ, $\sum G^{-q^k}$, G a polynomial in $\Gamma(x_1, \cdots, x_e)$ of $\deg G>0$ and $q>1$ a rational integer, the following result is proved. The series is algebraic if $q>1$ is of the form p^s, transcendental otherwise. (II) The character of the real numbers $\sum g^{-k^q}$, $k=0, 1, 2, \cdots$; $g>1$, $q>1$ rational integers, is not known. The corresponding series in Φ, $\sum G^{-k^q}$, is always transcendental. *A. J. Kempner* (Boulder, Colo.).

THEORY OF GROUPS

***van der Waerden, B. L. Die Gruppentheoretische Methode in der Quantenmechanik.** J. W. Edwards, Ann Arbor, Michigan, 1944. viii+157 pp. $6.00.

Reprint of volume 36 of the collection Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, published by J. Springer, Berlin. The original appeared in 1932.

Dirac, P. A. M. Unitary representations of the Lorentz group. Proc. Roy. Soc. London. Ser. A. **183**, 284–295 (1945). [MF 12049]

It is well known that if A_{rst} are the coefficients of a form $\sum A_{rst}\xi_1^r\xi_2^s\xi_3^t$ and the ξ are subjected to an orthogonal transformation then the quantities $(r!s!t!)^{\frac12}A_{rst}$ are transformed by an isomorphic orthogonal transformation. The author considers the A_{nrst} which are the coefficients of the form $\sum A_{nrst}\xi_0^{-n-1}\xi_1^r\xi_2^s\xi_3^t$ and subjects the ξ to a Lorentz transformation, that is, to a transformation which leaves $\xi_0^2-\xi_1^2-\xi_2^2-\xi_3^2$ invariant. He shows that the quantities $(r!s!t!/n!)^{\frac12}A_{nrst}$ are transformed by an orthogonal transformation. These transformations give a representation of the Lorentz group by unitary matrices of infinitely many dimensions (it is easy to show that, except for the trivial one, the Lorentz group has no finite unitary representation). Evidently it is sufficient to consider at the same time those A_{nrst} for which $r+s+t-n$ has a definite value; one obtains a representation for every value of $r+s+t-n$. This is true, according to the author, even if $r+s+t-n$ is not an integer and the representation obtained remains unitary if $r+s+t-n<1$. The quantities A_{nrst} are called expansors; if $r+s+t-n$ is constant, homogeneous expansors.

The author then associates with every expansor A_{nrst} a function $\sum A_{nrst}F_{nrst}(x_0, x_1, x_2, x_3)$, where the F_{nrst} are definite functions of the x, of integrable square, and chooses these in such a way that the scalar product of two expansors $\sum A_{nrst}B_{nrst}r!s!t!/n!$ is equal to the integral over the product of the corresponding functions. This can be done by considering an oscillator with the "energy" operator

$$\tfrac12(-x_0+x_1^2+x_2^2+x_3^2+\partial^2/\partial x_0-\partial^2/\partial x_1^2-\partial^2/\partial x_2^2-\partial^2/\partial x_3^2),$$

the characteristic states of which are the F_{nrst}.

The author finally considers a particle which is described not only by the space-time coordinates x_0, x_1, x_2, x_3 but also by four other coordinates and writes down a wave equation for such a particle. He concludes that the magnitude of the spin of such a particle depends on its state of motion. *E. P. Wigner* (Princeton, N. J.).

Miller, G. A. Groups involving a small number of sets of conjugate operators. Proc. Nat. Acad. Sci. U.S.A. **30**, 359–362 (1944). [MF 11367]

Let n denote the number of conjugate sets in a group G. If $n=3$, G is the group of order 3 or the noncyclic group of order 6. If $n=4$, G is one of the two groups of order 4 when it is Abelian and when it is non-Abelian it is either the dihedral group of order 10 or the tetrahedral group. The situation becomes complicated for $n\geqq 5$.
G. de B. Robinson (Ottawa, Ont.).

Kurosch, A. Isomorphisms of direct decompositions. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 185–202 (1943). (Russian. English summary) [MF 10183]

The author shows that to every group-theoretical theorem concerning the central isomorphism of direct decompositions there corresponds a parallel theorem on two-sided decompositions of rings into a direct sum without any assumptions about the commutativity or associativity of multiplication. To the centrum of the group corresponds the ideal $\mathfrak{N}$ of complete divisors of the null in the given ring, that is, the totality of elements a such that $ax=xa=0$ for any $x\subset R$. The central isomorphism is replaced by the $\mathfrak{N}$-isomorphism, that is, an isomorphism between two subrings of the ring R, for which the difference between the corresponding elements belongs to $\mathfrak{N}$.

This parallelism is caused by the fact that both theories can be regarded as special cases of the more general theory referring to complete Dedekind structures. By a complete Dedekind structure is meant a lattice in which (a) the sum and the product are defined for all the subsets of the lattice (finite and infinite) and (b) for any system of elements x_α, y_β satisfying the condition $x_\alpha\leqq y_\beta$ for $\alpha\neq\beta$ one has $(\sum x_\alpha)\prod y_\alpha=\sum x_\alpha y_\alpha$.

The application of general theorems about Dedekind structures to the theory of rings leads, for instance, to the following results. If the ring R does not contain any complete divisors of the null except the null itself, then any two direct decompositions of the ring possess a common continuation. Given any two two-sided direct decompositions of the ring R, there exist $\mathfrak{N}$-isomorphic continuations of these decompositions, provided the ideal $\mathfrak{N}$ is entirely contained in one of the elements of the given decompositions. *W. Hurewicz* (Cambridge, Mass.).

Halmos, Paul R. Comment on the real line. Bull. Amer. Math. Soc. **50**, 877–878 (1944). [MF 11556]

The author remarks that the additive group of the line is isomorphic (in the algebraic sense only) to the compact character group of the additive groups of rationals. Thus

it is possible to topologize the additive group of the line in such a way that it becomes compact. *P. A. Smith.*

Eckmann, Beno. Über monothetische Gruppen. Comment. Math. Helv. 16, 249–263 (1944).

The first part of the paper treats monothetic groups, that is, topological groups with a dense cyclic subgroup. The results are identical with results obtained by Halmos and Samelson [Proc. Nat. Acad. Sci. U. S. A. **28**, 254–258 (1942); these Rev. **4**, 2; evidently the author did not have access to this paper]; both papers make use of Pontrjagin's duality theory. In the second part the author defines equidistribution of a sequence x_i in a compact topological group G: let $N(M)$ be the number of those x_i's among the first N which are contained in the subset M of G; then the limit value of $N(M)/N$ has to equal the measure of M for every measurable M. Theorem: if the element a of the compact group G does not have 1 as eigenvalue in any nontrivial irreducible representation of G, then the powers of a are equidistributed. This is an extension of Weyl's theorem on equidistribution. For finite groups a simpler proof is given.
H. Samelson (Syracuse, N. Y.).

Malcev, A. On semi-simple subgroups of Lie groups. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 143–174 (1944). (Russian. English summary) [MF 11529]

The main problem of this paper is the determination of all classes of conjugate semi-simple subgroups of a semi-simple Lie group (connected, corresponding Lie algebra over the field of complex numbers). In §1 it is shown that, if $\mathfrak{G}$ is a general Lie group, then all its maximal semi-simple subgroups are conjugate and that the number of its classes of conjugate semi-simple subgroups is the same as that in a maximal semi-simple subgroup of $\mathfrak{G}$. It is also shown that the problems of classification of semi-simple groups, of solvable groups and of groups with a given radical and corresponding factor group are reducible to the main problem. Making use of the theory of representation this problem is further reduced to that of the enumeration of semi-simple subgroups of simple Lie groups. Thus consider the series of symplectic groups $\mathfrak{C}_n$; let $\mathfrak{G}$ be a given semi-simple group; it is required to enumerate, up to an equivalence, all subgroups of a given $\mathfrak{C}_p$ which are isomorphic with $\mathfrak{G}$. If $\mathfrak{H}$ is one such subgroup, the isomorphism $\mathfrak{G}\to\mathfrak{H}$ constitutes a representation of $\mathfrak{G}$ on $\mathfrak{C}_p$, that is, a symplectic representation of $\mathfrak{G}$ of degree $2p$. Conversely, every simplectic representation of $\mathfrak{G}$ of degree $2p$ determines uniquely a subgroup of $\mathfrak{C}_p$.

Now every representation is uniquely determined by a linear form Λ, the weight of the representation, and the weights for all semi-simple groups are known. Consequently, for the solution of our problem we must select from the complete series of weights $\Lambda_1, \Lambda_2, \cdots$ of $\mathfrak{G}$ those which correspond to symplectic representations of $\mathfrak{G}$. Such a selection is made in §2 for symplectic as well as for orthogonal representations. Knowing the weights $\Lambda_{\alpha_1}, \Lambda_{\alpha_2}, \cdots$ of all symplectic representations of $\mathfrak{G}$ we know all the subgroups isomorphic to $\mathfrak{G}$ of the whole set of groups $\mathfrak{C}_n$. To determine the subgroups isomorphic to $\mathfrak{G}$ of some particular $\mathfrak{C}_p$ we need only use Weyl's method to determine the degrees of the representations with weights Λ_{α_i} and select those whose degree is $2p$. In this manner the main problem is solved in §2 for the general simple groups $\mathfrak{A}_n, \mathfrak{B}_n, \mathfrak{C}_n, \mathfrak{D}_n$. To carry out this solution for the special simple groups the author first proves that, if $\{e, f, e^*\}$ and $\{e_1, f, e_1^*\}$ are two simple subgroups having an element in common, when $\{e, f, e^*\}$ are in canonical form, that is, $[fe]=e$, $[fe^*]=-e^*$, $[ee^*]=f$, with similar relations for e_1, fe_1^*, then the two subgroups are conjugate in any semi-simple subgroup containing them. By means of this result the author obtains the explicit enumeration of all semi-simple subgroups of the two special simple groups $\mathfrak{G}_2$ and $\mathfrak{F}_4$. The problem for the three special simple groups $\mathfrak{E}_6$, $\mathfrak{E}_7$, $\mathfrak{E}_8$ is reduced to the study of a large number of combinations of root-vectors, which is not carried out in this paper.
M. S. Knebelman (Pullman, Wash.).

Ado, I. D. On nilpotent algebras and p-groups. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 299–301 (1943). [MF 11177]

An infinite p-group (nil-ring) is said to be locally finite (locally nilpotent) if every finite set of elements generates a finite subgroup (nilpotent subring). It is known that, for any nil-ring N, the set of elements $E+r$, where $r\epsilon N$ and E is a unit element, forms a group G under multiplication. The author proves that, if N is a locally nilpotent algebra of characteristic p, then G is a locally finite p-group. Conversely, it is shown that any locally finite p-group G determines a locally nilpotent algebra over any field of characteristic p. This algebra N is shown to have the following property: the group H of all elements of the form $E+r$, where $r\epsilon N$, contains a subgroup $\bar{G}\cong G$, such that every proper ideal of N determines a proper normal subgroup of $\bar{G}$. The above results are extensions of those announced by the reviewer [Bull. Amer. Math. Soc. **46**, 229 (1940)]. Applications are given of the above theory to prove the existence, for every prime p, of a locally finite p-group which coincides with its derived group, and to prove that a locally finite p-group which has no center contains an infinite descending chain of normal subgroups.
S. A. Jennings (Vancouver, B. C.).

Brandt, Angeline. The free Lie ring and Lie representations of the full linear group. Trans. Amer. Math. Soc. **56**, 528–536 (1944). [MF 11495]

This paper forms a continuation of work of R. M. Thrall [Amer. J. Math. **64**, 371–388 (1942); these Rev. **3**, 262]. The free Lie ring, its characteristic ideals and certain related right ideals in the group ring of the symmetric group are studied. Thrall had given a recursion formula for the character $[m]$ of the mth Lie representation. The main result of the present paper is the following explicit formula for this character: $[m]=(1/m)\sum\mu(d)s_d^{m/d}$, where d ranges over all divisors of m, $\mu(d)$ is the Möbius function and s_d is the sum of the dth powers of the characteristic roots of an arbitrary element A of the full linear group. If A is the unit matrix, this gives a result of E. Witt [J. Reine Angew. Math. **177**, 152–160 (1937)] for the dimension of the module of the homogeneous expressions of dimension m in the free Lie ring.
R. Brauer (Toronto, Ont.).

Thrall, R. M. On the decomposition of modular tensors. II. Ann. of Math. (2) **45**, 639–657 (1944). [MF 11373]

In part I [same Ann. (2) **43**, 671–684 (1942); these Rev. **4**, 134] the reduced form of the mth power representation π_m of the full linear group $G(n, \mathfrak{k})$ over a modular field $\mathfrak{k}$ of order q was determined subject to the restriction that $m<2p$ and $m\leqq q$. Here the author proves that every irreducible modular representation of G is equivalent to a tensor representation; also, if q is finite and Γ is the $\mathfrak{k}$-group ring of G, there exists a faithful tensor representation of Γ.

If $q=p$ the reduction of π_m given in part I must be modified according as (i) $p\leq m<2p-1$; (ii) $m=2p-1$, $n>2$; (iii) $m=2p-1$, $n=2$. *G. de B. Robinson* (Ottawa, Ont.).

Gelfand, I., and Raikov, D. Irreducible unitary representations of locally bicompact groups. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 199–201 (1944). [MF 11633]

Gelfand, I., and Raikov, D. Irreducible unitary representations of locally bicompact groups. Rec. Math. [Mat. Sbornik] N.S. **13**(55), 301–316 (1943). (Russian. English summary) [MF 11655]

If G is a topological group, a unitary representation of G is a continuous homomorphism of G into the group of unitary transformations of a Hilbert space (possibly nonseparable). Such a representation is said to be reducible if there exists a nontrivial projection in the Hilbert space which commutes with every transformation of the representation. The principal result is that, if G is locally compact and $g\epsilon G$ is not the identity, there exists an irreducible unitary representation under which the image of g is not the identity. This extends to locally compact groups a result already known for compact groups [J. von Neumann, Ann. of Math. (2) **34**, 170–190 (1933)]. No statement is made concerning the possibility of decomposing (in some sense) a representation into irreducible parts.
N. E. Steenrod (Ann Arbor, Mich.).

Hurevitsch, Anna. Unitary representation in Hilbert space of a compact topological group. Rec. Math. [Mat. Sbornik] N.S. **13**(55), 79–86 (1943). (English. Russian summary) [MF 11647]

This paper gives an analysis of representations of a compact group by means of unitary operators in Hilbert space. The author begins by defining an integral for functions defined on a compact group with values in Hilbert space. With this as a tool the following theorem is proved. "If there is an arbitrary unitary representation U_s of a compact topological group G with the second countability axiom in Hilbert space H, then H can be decomposed into the direct sum of enumerably many mutually orthogonal finite dimensional subspaces invariant and irreducible under the group of operators U_s." Certain sums of these spaces are defined and these yield a sequence of invariant (not necessarily finite dimensional) subspaces which is shown to be uniquely determined. [For closely related results, see the paper by S. Bochner and J. von Neumann, Trans. Amer. Math. Soc. **37**, 21–50 (1935), especially theorem 39.] *D. Montgomery* (Princeton, N. J.).

Janet, Maurice. Sur les formules fondamentales de la théorie des groupes continus finis et de la méthode du repère mobile. Ann. Sci. École Norm. Sup. (3) **59**, 165–186 (1942). [MF 11882]

The paper is largely expository in character. It gives a treatment of the local theory of Lie groups based on É. Cartan's method of moving frames and will be found useful for the understanding of Cartan's theory. A formula previously given by the author [C. R. Acad. Sci. Paris **212**, 424–425 (1941); these Rev. **3**, 35] plays an important rôle here. *S. Chern* (Princeton, N. J.).

Bruck, R. H. Simple quasigroups. Bull. Amer. Math. Soc. **50**, 769–781 (1944). [MF 11294]

A quasigroup with unit element is called a loop. A quasigroup is said to be simple if it has no proper homomorph. A proof is given of a conjecture due to A. A. Albert that there exist simple loops of every finite order other than 4. It is further shown that for every integer $m>1$ there exists a simple loop G, of order $2m+1$, containing as subloop an arbitrary loop F of order m, and such that every proper subloop of G is contained in F. The proofs of both these theorems are constructive. A quasigroup extension $P=(H,Q)$ of an arbitrary set H by a quasigroup Q is defined. Every nonsimple quasigroup G is such an extension, where Q is isomorphic to a proper homomorph of G. It is shown that every loop isotopic to P is isomorphic to an extension $P_0=(H_0, Q_0)$ of a set H_0 of the same order as H (and which in fact may be taken to be a loop) by a loop Q_0 isotopic to Q. It follows that every isotope of a simple loop is simple.
D. C. Murdoch (Vancouver, B. C.).

Smiley, M. F. An application of lattice theory to quasigroups. Bull. Amer. Math. Soc. **50**, 782–786 (1944). [MF 11295]

The paper contains an application of the reviewer's formulation of a Jordan-Hölder theorem for arbitrary partially ordered sets to give a proof of this theorem for quasi-groups and loops as studied recently by A. A. Albert. The author supplements the theory of loops with a number of properties of normal subloops. *O. Ore.*

Mann, Henry B. On certain systems which are almost groups. Bull. Amer. Math. Soc. **50**, 879–881 (1944). [MF 11557]

A "left right" system $\mathfrak{S}$ is a set of elements $A, B, C, \cdots$ which satisfy the following postulates. (1) To every ordered pair of elements A, B of $\mathfrak{S}$ there exists a unique product AB which is an element of $\mathfrak{S}$. (2) $(AB)C=A(BC)$. (3) $\mathfrak{S}$ contains a left unit E (not necessarily unique). (4) Every element A has a right inverse A' such that $AA'=E$. A left right system is idempotent if each of its elements is idempotent. There exists one and only one idempotent left right system of any given order n, and this consists of elements $A_1, \cdots, A_n$ with the law of multiplication $A_iA_j=A_j$. Finally, every left right system is the direct product of an idempotent left right system and a group. If $S_1, \cdots, S_n$ are n statements and if A_i is the statement "the statements $S_1, \cdots, S_{i-1}$ are annulled but S_i is true," then the statements $A_1, \cdots, A_n$ form an idempotent left right system. *D. C. Murdoch* (Vancouver, B. C.).

ANALYSIS

Bronowski, J. An inequality relating means. Proc. Cambridge Philos. Soc. **40**, 253–255 (1944). [MF 11872]

Let a and b be positive numbers such that b lies between a and 1; let $k_1, \cdots, k_n$ be real numbers, not all equal, and let k be their arithmetic mean. Then

$$\sum a^{k_i}/a^k > \sum b^{k_i}/b^k.$$

Several proofs are given. *R. P. Boas, Jr.*

Boas, R. P., Jr. A differential inequality. Bull. Amer. Math. Soc. **51**, 95–96 (1945). [MF 11822]

If $f'(x)$ is absolutely continuous in $(0, a)$, $f(0)=f'(0)=0$ and

$$f''(x)\leq K(x)|f'(x)|+x^{-1}L(x)|f(x)|$$

almost everywhere in $(0, a)$, where $K(x)$ and $L(x)$ are nonnegative and integrable in $(0, a)$, then either $f(x)\equiv 0$ in

some interval $(0, b)$ or $f'(x)<0$ in $0<x<\min(a, c)$, where c is such that

$$\int_0^x \{K(t)+L(t)\}dt<1, \qquad 0<x<c.$$

A. Zygmund (South Hadley, Mass.).

Roussel, André. Sur la détermination des fonctions par leur accroissement infinitésimal. Bull. Soc. Math. France **70**, 1–30 (1942). [MF 11886]

The author studies the determination of $f(x)$ from

$$f(x+h)-f(x)=g(x, h)+\epsilon\omega(h),$$

where $g(x, h)$ is a given continuous function, and it is known that ϵ approaches zero with h, and $\omega(h)/h^m$ approaches zero with h for a known positive m. He finds a condition for the existence of $f(x)$ in terms of sums, and also one in terms of double integrals. If $m\geqq 1$, $f(x)$ is uniquely determined to within an additive constant, but, if $m<1$, $f(x)$ may be modified by a function of restricted type. *P. Franklin* (Cambridge, Mass.).

Levin, V. On an integral analogue of MacLaurin series. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 51–52 (1944). [MF 11624]

The integral studied is obtained from MacLaurin's series by replacing factorials, derivatives and summation by gamma functions, Riemann-Liouville fractional derivatives and integration, respectively. For sufficiently regular functions, this integral equals the original function together with a correction term expressed as an integral. The explicit application to several elementary functions is carried out in detail. *P. Franklin* (Cambridge, Mass.).

Westberg, R. On the integral theorems of Gauss and Stokes. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **13**, no. 15, 10 pp. (1943). [MF 11540]

The author derives generalizations of the Gauss and Stokes integral theorems. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ denote the unit vectors along the rectangular x, y, z axes, and let ∇ denote the vector nabla operator. Further, let $\mathbf{u}$ denote a vector function of position, $\mathbf{p}$ a tensor function of position, $P\{\mathbf{u}, \mathbf{p}\}$ a function of position which is one-valued, continuous, possesses continuous first derivatives and is linear in $\mathbf{u}$. Denoting the outward-drawn vectorial element of area of a surface F, bounding some domain V, by $d\mathbf{f}$, the author derives the generalized Gauss theorem

$$\int_V P\{\nabla, \mathbf{p}\}dV=\int_F P\{d\mathbf{f}, \mathbf{p}\}.$$

Next, writing the vector product as $(\mathbf{u}\times\mathbf{w})$ and the vectorial element of arc length of a curve S, bounding a surface F, as $d\mathbf{s}$, the author obtains the generalized Stokes theorem

$$\int_F P\{d\mathbf{f}\times\nabla, \mathbf{p}\}=\int_S P\{d\mathbf{s}, \mathbf{p}\}.$$

The two dimensional forms of these theorems are determined. Several examples are studied. *N. Coburn.*

Menger, Karl. Methods of presenting e and π. Amer. Math. Monthly **52**, 28–33 (1945). [MF 11908]

Theory of Functions of Complex Variables

✶Hurwitz, Adolf, und Courant, R. Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen. Interscience Publishers, Inc., New York, 1944. xii+534 pp. $7.50.

Reprint of the third (latest) German edition of volume 3 of the collection: Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. The original was published by J. Springer, Berlin, 1929.

Erdös, P. Note on the converse of Fabry's gap theorem. Trans. Amer. Math. Soc. **57**, 102–104 (1945). [MF 11912]

The author gives an elementary proof of the following theorem of Pólya: let n_k be a sequence of integers for which $\liminf n_k/k<\infty$; then there exists a power series $\sum a_k z^{n_k}$ whose circle of convergence is the unit circle and for which the unit circle is not a natural boundary. According to Fabry's theorem this is no longer true if $\lim n_k/k=\infty$.

H. Pollard (New York, N. Y.).

Duffin, R. J., and Schaeffer, A. C. Power series with bounded coefficients. Amer. J. Math. **67**, 141–154 (1945). [MF 11929]

Let $f(z)$ be regular and satisfy $|f(z)|<Ae^{k|z|}$, $k<\pi$, in $\Re(z)\geqq 0$. Let (*) $|\lambda_n-n|\leqq\Gamma\ (n>\Gamma)$ and $|\lambda_n-\lambda_m|\geqq\gamma>0$ for $n\neq m$. The authors prove that, if $\{f(\lambda_n)\}$ is a bounded sequence, $f(z)$ is bounded on the positive real axis; a more precise result holds if $f(z)$ is entire and bounded at points λ_n satisfying (*) for all positive and negative n. These results are considerably stronger than previous results of similar character. The authors also prove that $f(z)$ is bounded on the real axis if $\{\lambda_n\}$ contains all the positive and negative integers except for a sequence $\{\mu_n\}$ such that $\mu_{n+1}-\mu_n\to\infty$ as $|n|\to\infty$.

As an application, the authors prove that a function $\sum_0^\infty a_n z^n$, where the a_n take only a finite number of different values, is a rational function if it is bounded in a sector of the unit circle. This generalizes a result of Szegö [Math. Ann. **87**, 90–111 (1922)], in which the conclusion is that $f(z)$ is rational if it can be continued analytically beyond the unit circle. Another result is that, if $\sum_0^\infty a_n z^n$ is bounded in a sector of the unit circle and the a_n are bounded except perhaps when $n=n_j$, $n_{j+1}-n_j\to\infty$, then all the a_n are bounded. *R. P. Boas, Jr.* (Cambridge, Mass.).

Maitland, B. J. The flat regions of integral functions of finite order. Quart. J. Math., Oxford Ser. **15**, 84–96 (1944). [MF 11804]

Suppose that $f(z)$ is an entire function of order two and mean type κ. It is shown that there is a sequence of circles of the same diameter d which depends on κ, with centers tending to infinity, and with the property that, for every $\epsilon>0$, $\log|f(z)|>(\kappa-\epsilon)|z|^2$ over all but a finite number of the circles. Thus the minimum modulus of the function over these circles is in a sense of the same rate of growth as the maximum modulus of the function over the entire plane. The theorem is also generalized to functions which are regular in an angle, and to entire functions of order other than two. In these cases the circles are not of uniform diameter. These results are obtained using an interpolation formula which the author has studied in a previous paper [Proc. London Math. Soc. (2) **45**, 440–457 (1939); these Rev. **1**, 49]. *A. C. Schaeffer.*

Dufresnoy, Jacques. Sur quelques propriétés des cercles de remplissage des fonctions méromorphes. Ann. Sci. École Norm. Sup. (3) 59, 187–209 (1942). [MF 11883]

The author here proves a number of results previously announced [C. R. Acad. Sci. Paris 214, 467–469 (1942); these Rev. 4, 138]. The theorems obtained are in the direction of the Nevanlinna theory of meromorphic functions. The principal tool used is the notion of cercles de remplissage in the sense of Valiron-Milloux: the meromorphic function $f(z)$ is said to have the set of cercles de remplissage Γ_1, Γ_2, $\cdots$ (where Γ_k is the circle of radius r_k about z_k) if both $|z_k|$ and $|z_k|/r_k$ increase indefinitely with k and if $f(z)$ assumes in Γ_k all values of the complex sphere except possibly for some which lie within two circles of a spherical radius tending to zero with $1/k$. *O. Helmer.*

Schaginjan, A. L. Remarques sur l'étude des approximations au moyen des fonctions rationnelles dans le domaine complexe. C. R. (Doklady) Acad. Sci. URSS (N.S.) 44, 47–51 (1944). [MF 11605]

A dozen theorems are stated, dealing with extensions of results by S. N. Bernstein and M. A. Lavrentieff on the approximation to functions by means of polynomials. A typical result is the following. Let D be any simply connected domain lying in the interior of the strip $|y|\leqq a$ $(z=x+iy)$ and topologically equivalent to it. If $f(z)$ is regular inside D, is continuous in D plus its boundary and satisfies in D the condition $\lim f(z)e^{-|z|}=0$ as $|z|\to\infty$, then there exists a sequence of polynomials $\{P_n(z)\}$ such that, in D,

$$\sup \exp(-|z|^{1+\epsilon})|f(z)-P_n(z)|\to 0, \qquad n\to\infty,$$

for every $\epsilon>0$. Moreover there are functions for which the exponent $1+\epsilon$ cannot be replaced by any number $1-\epsilon$ $(\epsilon>0)$. Proofs are not offered, but a method is suggested at the end. *I. M. Sheffer* (State College, Pa.).

van Aardenne-Ehrenfest, T., und Wolff, Julius. Über die Grenzen der einfachzusammenhängenden Gebiete. Comment. Math. Helv. 16, 321–323 (1944).

The following theorem is established. If the boundary of a simply-connected region G has only prime ends of the first and second kind, then the set m of prime ends of the first kind has the power of the continuum. Under a (1,1) conformal mapping of G onto a half-plane, the set m corresponds to a set of the second category relative to the boundary line of the half-plane. The proof is based upon anterior results of Wolff [Nederl. Akad. Wetensch., Proc. 45, 169–170 (1942); these Rev. 6, 61] concerning the behavior of a mapping function of a simply-connected region in the neighborhood of the boundary. The theorem of the present note yields a negative answer to the question of Carathéodory concerning the existence of a simply-connected region whose boundary consists exclusively of prime ends of the second kind. *M. H. Heins.*

Chvedelidze, B. V. On a linear boundary value problem of Riemann for a system of analytic functions. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 289–296 (1943). (Russian. Georgian summary) [MF 11702]

The author considers the problem of finding functions $f_k(z)$ $(k=1,\cdots,n)$, analytic in a simply connected domain, such that (i) their N derivatives have boundary values satisfying a Hölder condition and (ii) these boundary values satisfy the following system of integral equations

$$\sum_{\alpha=0}^{N}\sum_{\beta=1}^{n}\Re\left\{a_{\alpha\beta\gamma}(t)f_\beta^{(\alpha)}(t)+\int_L h_{\alpha\beta\gamma}(t,t_1)f_\beta^{(\alpha)}(t_1)ds_1\right\}=C_\gamma(t),$$

where $\gamma=1,\cdots,n$ and $a_{\alpha\beta\gamma}$, $h_{\alpha\beta\gamma}$, C_γ are functions given on the curve L enclosing the domain. Furthermore, $a_{\alpha\beta\gamma}$ and C_γ satisfy Hölder's condition, and

$$h_{\alpha\beta\gamma}=h^*_{\alpha\beta\gamma}(t,t_1)/|t-t_1|^\lambda$$

with $\lambda<1$ and $h^*_{\alpha\beta\gamma}$ satisfying Hölder's condition.

The author uses ideas of I. N. Vekua [Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 11, 109–139 (1942); 12, 215 (1943); these Rev. 6, 123] and N. I. Muschelišvili [Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 987–994 (1942); these Rev. 5, 269]. He succeeds in reducing the problem to the solution of a system of singular equations and deduces necessary and sufficient conditions for the existence of a solution of the given problem. The result is applied to the special case where $h_{\alpha\beta\gamma}=0$. *František Wolf* (Berkeley, Calif.).

Theory of Series

Hamming, R. W. Convergent monotone series. Amer. Math. Monthly 52, 70–72 (1945). [MF 11904]

Certain theorems having to do with "slowness" of convergence and divergence which are now classical in the theory of series of positive terms are extended in this paper in the sense that classical conclusions are replaced by more extreme results but only after adding hypotheses. These hypotheses have to do with the derived series $\sum 2^n a_{2^n}$. *T. Fort* (Bethlehem, Pa.).

Rajagopal, C. T. The Abel-Dini and allied theorems. Amer. Math. Monthly 51, 566–570 (1944). [MF 11733]

According to the author, his observations are "in the nature of a postscript to two papers" [T. H. Hildebrandt, same Monthly 49, 441–445 (1942); C. T. Rajagopal, same Monthly 48, 180–185 (1941); these Rev. 4, 79; 2, 277]. "Our familiar convergence theorems as well as canonical forms for the criteria of convergence can be shown to have their origin in Maclaurin's method of condensing a series by means of an integral. This point of view helps us to achieve a pedagogically satisfactory 'systematization of the general theory of convergence.' " In addition to emphasizing this point of view the author proves several theorems on convergence which are generalizations of previously known results. *T. Fort* (Bethlehem, Pa.).

Hardy, G. H. Note on the multiplication of series by Cauchy's rule. Proc. Cambridge Philos. Soc. 40, 251–252 (1944). [MF 11871]

A short and elementary proof of the following theorem is given. If $\sum a_n$ and $\sum b_n$ are convergent to A and B, if $x=y+z$, where $y(x)$ and $z(x)$ increase steadily to infinity with x, and if

$$\sum_{n=y}^{x}|a_n|=O(1), \quad \sum_{n=z}^{x}|b_n|=O(1),$$

then the Cauchy product series converges to AB. The "symmetric" theorem obtained by setting $y=z=\frac{1}{2}x$ was proved by L. Neder [Proc. London Math. Soc. (2) 23, 172–184 (1924)].

[It is interesting to note that we can delete the hypothesis that $y(x)$ and $z(x)$ increase steadily to infinity with x. In case $\liminf y(x) < \infty$ or $\liminf z(x) < \infty$, one of $\sum a_n$ and $\sum b_n$ converges absolutely and the classic theorem of Mertens [J. Reine Angew. Math. **79**, 182–184 (1874)] guarantees convergence of the Cuachy product to AB. In case $y(x) \to \infty$ and $z(x) \to \infty$ (but not necessarily monotonely), Hardy's proof applies verbatim.]

R. P. Agnew (Ithaca, N. Y.).

Broudno, A. L. Sommation des suites bornées par les méthodes linéaires régulières. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 183–185 (1944). [MF 11615]

The note states (without proofs) a number of results concerning summation of bounded sequences by Toeplitz matrices. If $A = (a_{ik})$ is a Toeplitz matrix, the set Ω of all bounded sequences $\{S_n\}$ summable A is called the bounded field of A. The following theorems will indicate the character of the results. (i) Let A and B be two Toeplitz matrices, and suppose that every bounded sequence summable A is also summable B. Then the B-limits of such sequences are the same as the A-limits. (ii) Suppose that Ω_a and Ω_b are two bounded fields such that $\Omega_a \subset \Omega_b$ (here $\subset$ means strict inclusion). Then it is possible to make correspond to every number ν, $0 \leqq \nu \leqq 1$, a bounded field Ω_ν such that (1) $\Omega_a \subseteq \Omega_0 \subset \Omega_1 = \Omega_b$; (2) if $0 \leqq \nu < \mu \leqq 1$, then $\Omega_\nu \subset \Omega_\mu$; (3) if $0 \leqq \nu \leqq \mu_n \leqq 1$ for all n, and if $\lim \mu_n = \nu$, then $\Omega_\nu = \lim \Omega_{\mu_n}$.

A. Zygmund (South Hadley, Mass.).

Allen, H. S. *T*-transformations which leave the core of every bounded sequence invariant. J. London Math. Soc. **19**, 42–46 (1944). [MF 11320]

The core (Kern, defined by Knopp) of a bounded complex sequence is the least convex set containing the limit points of the sequence. In order that a regular transformation $y_n = \sum a(n, k)x_k$ be such that the core Γ of the transform y_n of each bounded complex sequence x_n is identical with the core C of x_n, it is necessary and sufficient that $\sum_{k=1}^{\infty} |a(n, k)| \to 1$ as $n \to \infty$ and that, for each increasing sequence k_i of integers, the number 1 is a limit point of the sequence $u_n = \sum_{i=1}^{\infty} a(n, k_i)$. The conditions imply that the set of limit points of each bounded sequence x_n is included in the set of limit points of the transform y_n. Similar results, not cited in the paper, have been obtained by Raff [Math. Z. **37**, 572–577 (1933)] and Winn [C. R. Acad. Sci. Paris **196**, 154–156 (1933)].

R. P. Agnew (Ithaca, N. Y.).

Agnew, Ralph Palmer. A genesis for Cesàro methods. Bull. Amer. Math. Soc. **51**, 90–94 (1945). [MF 11821]

The author proves the following theorem. The Cesàro methods are the only methods of summability, regular or not, which are both Nörlund methods and Hurwitz-Silverman-Hausdorff methods. He also shows that the only methods which are both Riesz methods and Hurwitz-Silverman-Hausdorff methods are methods Γ_r closely related to the methods C_r.

T. Fort (Bethlehem, Pa.).

Agnew, Ralph Palmer. Convergence fields of methods of summability. Ann. of Math. (2) **46**, 93–101 (1945). [MF 11792]

"The convergence field of a method of summability is the set of sequences summable by the method. This paper is concerned with bounded sequences in the convergence fields of multiplicative methods of summability." What the author does is to prove several sufficient conditions that a sequence belong to a particular field. He shows that if either $1, -1, 1, -1, \cdots$ or $1, 0, 1, 0, \cdots$ belongs to a field then so does the other and that the field includes "many" additional sequences.

T. Fort (Bethlehem, Pa.).

Hill, J. D. Nörlund methods of summability that include the Cesàro methods of all positive orders. Amer. J. Math. **67**, 94–98 (1945). [MF 11925]

A complex sequence p_n for which $P_n = p_0 + \cdots + p_n \neq 0$ for each n determines a Nörlund method N_p of summability by means of which a sequence s_n is summable to s if

$$t_n = (p_n s_0 + p_{n-1} s_1 + \cdots + p_0 s_n)/P_n \to s$$

as $n \to \infty$. Necessary and sufficient conditions for regularity of $N_p C_r^{-1}$ are given; these are conditions that N_p include C_r. Simpler sufficient conditions are given and used to show that the Nörlund method for which $p_n = \cosh n^{\frac{1}{2}}$ includes C_r for each $r > 0$ (and hence also for each complex r for which $\Re(r) > -1$).

R. P. Agnew (Ithaca, N. Y.).

Wintner, Aurel. A summation method associated with Dirichlet's divisor problem. Amer. J. Math. **66**, 579–590 (1944). [MF 11396]

The transformation

$$D_n(s) = \sum_{m=1}^{n} (n/m - [n/m]) s_m / n$$

of the sequence $s_1, s_2, \cdots$ into the sequence $D_1(s), D_2(s), \cdots$ defines the summation method considered. It is compared with $(C, 1)$ summation, defined by $M_n(s) = \sum_{m=1}^{n} s_m / n$. These two methods are shown to be incomparable but not inconsistent and other properties of the D-process are discussed. The presentation is marred by a number of misprints and errors.

H. S. Zuckerman (Seattle, Wash.).

Tchélidzé, V. Le théorème d'Abel pour une série double. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **4**, 201–206 (1943). (Russian. Georgian and French summaries) [MF 11698]

Nous démontrons dans cette note le théorème suivant: si la série double $S = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l}$ converge et ses sommes partielles $S_{m,n} = \sum_{k=0}^{m} \sum_{l=0}^{n} a_{k,l}$ satisfont à la condition $|S_{m,r}| \leqq A(m+1)^{\alpha}(n+1)^{\alpha}$, $\alpha < 1$ et A étant des nombres fixes, la série double de puissances

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} x^k y^l$$

converge absolument pour $|x| < 1$, $|y| < 1$. De plus,

$$\lim_{(x,y)_\lambda \to 1} f(x, y) = S,$$

où le passage à la limite doit être effectuée de manière que

$$(1) \qquad \frac{1}{\lambda} \leqq \frac{1-x}{1-y} \leqq \lambda,$$

$\lambda \geqq 1$ un nombre quelconque fixe. Il est à remarquer que J. C. Vignaux [Bol. Un. Mat. Ital. **17**, 209–214 (1938)] à démontré ce théorème dans le cas où la suite double de sommes partielles est bornée, toutefois sans recourir à la condition (1).

From the author's summary.

Bissinger, B. H. A generalization of continued fractions. Bull. Amer. Math. Soc. **50**, 868–876 (1944). [MF 11555]

The regular continued fraction

$$\frac{1}{a_1+} \frac{1}{a_2+} \cdots$$

may be thought of as the result of continued iteration of the function $f(t)=1/t$ in the form (1) $f(a_1+f(a_2+\cdots))$, where the a's are positive integers. This paper studies the properties of the infinite process (1) when f is allowed to be any member of a class F or, in particular, F_p. Here F is the class of all monotone strictly decreasing functions $f(t)$ defined for $t>1$, with $f(1)=1$ and $f(\infty)=0$, and such that the slope of the line joining any two points on the graph of $y=f(x)$ is greater than -1, being in fact bounded away from -1 for $x>f(2)+1$. The class F_p is that subclass of F in which $f(t)$ is linear between integer values of t. The number x being given between 0 and 1, there is an algorithm (generalizing that of Euclid) for obtaining a set of corresponding a's as follows: $z_0=f^{-1}(x)$, $a_1=[z_0]$, $\theta_1=z_0-a_1$; $z_n=f^{-1}(\theta_n)$, $a_{n+1}=[z_n]$, $\theta_{n+1}=z_n-a_{n+1}$; f^{-1} is the inverse function of f. The truncated expressions $x_1=f(a_1)$, $x_2=f(a_1+f(a_2))$, $x_3=f(a_1+f(a_2+f(a_3)))$, $\cdots$ are called the successive convergents of x. These converge to x with errors alternately by defect and excess as in the case of the continued fraction.

If f belongs to the subclass F_p the following facts are proved. As functions of x, the a's are "statistically independent." The set of x for which the a's are bounded is of measure zero, as is the set for which all the a's exceed unity. The great generality of the class F restricts the results of the paper to the set-theoretic type. *D. H. Lehmer.*

Wall, H. S., and Wetzel, Marion. Contributions to the analytic theory of *J*-fractions. Trans. Amer. Math. Soc. 55, 373–392 (1944). [MF 10498]

This paper is concerned with complex continued fractions of the form (*J*-fractions)

$$\frac{1}{b_1+z-}\,\frac{a_1^2}{b_2+z-}\,\frac{a_2^2}{b_3+z-}\cdots.$$

In an earlier paper [Ann. of Math. (2) 44, 103–127 (1943); these Rev. 4, 244] Hellinger and Wall treated the case where a_p is real and the imaginary part of b_p is nonnegative. These conditions imply that each quadratic form

$$\sum_{r=1}^{p}\mathfrak{J}(b_r+z)\xi_r^2-2\sum_{r=1}^{p-1}\mathfrak{J}(a_r)\xi_r\xi_{r+1},\quad p=1,2,3,\cdots,$$

is positive definite in the real variables ξ_r when $\mathfrak{J}(z)>0$. The present analysis is an extension of the earlier investigation; *J*-fractions are studied under the more general assumption that the corresponding quadratic form is positive definite for $\mathfrak{J}(z)>0$. Results analogous to those in the earlier paper are obtained. *W. Leighton.*

Wall, H. S., and Wetzel, Marion. Quadratic forms and convergence regions for continued fractions. Duke Math. J. 11, 89–102 (1944). [MF 10150]

The authors continue their study of the continued fraction

$$(1)\qquad \frac{1}{b_1+z-}\,\frac{a_1^2}{b_2+z-}\,\frac{a_2^2}{b_3+z-}\cdots$$

and its associated quadratic form

$$(2)\qquad \sum_{p=1}^{\infty}(b_p+z)x_p^2-2\sum_{p=1}^{\infty}a_px_px_{p+1},$$

where the numbers a_p, b_p and z are complex and x_p are real ($p=1, 2, 3, \cdots$). Denoting the imaginary parts of a_p, b_p and z by α_p, β_p and y, respectively, they show that a necessary and sufficient condition that (2) be positive definite for $y>0$ is that $\beta_p\geqq 0$ and $\alpha_p^2=\beta_p\beta_{p+1}(1-g_{p-1})g_p$ ($p=1, 2, 3, \cdots$), where g_i are any real numbers satisfying the conditions $0\leqq g_i\leqq 1$ ($i=0, 1, 2, \cdots$). The authors then show that, if $\liminf|a_p|<\infty$, a positive definite continued fraction (1) converges uniformly in every bounded closed region in the upper half-plane $y>0$. It is shown that this result includes a number of known general theorems on convergence of continued fractions. *W. Leighton.*

Wall, H. S. Note on the expansion of a power series into a continued fraction. Bull. Amer. Math. Soc. 51, 97–105 (1945). [MF 11823]

The author gives an algorithm for determining a continued fraction

$$\frac{a_0}{b_1+z-}\,\frac{a_1}{b_2+z-}\,\frac{a_2}{b_3+z-}\cdots$$

whose nth approximant, when expanded in descending powers of z, gives the first $2n$ terms of the power series $\sum_0^\infty c_nz^{-n-1}$. He shows that the problem of determining such an algorithm is equivalent to the problem of constructing a sequence of polynomials $\{P_n\}$ orthogonal with respect to the moment operator S which operates on a polynomial by replacing z^p by c_p; P_n and P_m are orthogonal if $S(P_nP_m)=0$ ($n\neq m$). *R. P. Boas, Jr.* (Cambridge, Mass.).

Special Functions

Robbins, Herbert E. Two properties of the function cos *x*. Bull. Amer. Math. Soc. 50, 750–752 (1944). [MF 11290]

It is shown that, if $f(x)$ is continuous and periodic with period 2π and if there exist constants a and b such that $f(x+1)+f(x)\equiv af(x+b)$, then $f(x)$ is necessarily a cosine function, that is, there exist constants A, B, n (the last an integer) such that $f(x)\equiv A\cos n(x+B)$. A similar theorem is proved in case $f(x)$ is continuous, differentiable, periodic, and such that constants α and β exist with $f'(x)\equiv\alpha f(x+\beta)$. The former result is applied to derive from certain basic assumptions the parallelogram law for the addition of forces. *D. C. Lewis* (New York, N. Y.).

Jonah, Harold F. S. Development of certain quadratic functional equations. Bull. Amer. Math. Soc. 51, 147–151 (1945). [MF 11831]

W. Maier [Math. Ann. 104, 745–769 (1931)] put

$$f(x,u)=\sum_{r=-\infty}^{+\infty}\frac{e^{2\pi i x r}}{u+r},$$

where $0<x<1$, and u is not an integer. He proved the functional equation

$$f(x,u)f(\xi,v)=f(\xi,u+v)f(x-\xi,u)-f(x,u+v)f(x-\xi,-v),$$

where $0<\xi<x<1$, and none of u, v, $u+v$ is integral. Jonah gives the formula

$$f(x,u)=2\pi ie^{-2\pi ixu}/(1-e^{-2\pi iu}),$$

which is valid for all x. Next he defines

$$f_p(x,u)=f(x+p,u)-f(x,u),$$

where p is integral, and derives four addition theorems for this function. *L. Carlitz* (Durham, N. C.).

Truesdell, C. On a function which occurs in the theory of the structure of polymers. Ann. of Math. (2) 46, 144–157 (1945). [MF 11796]

Several different expressions are derived for the Dirichlet series $\phi(x, s)=\sum_{n=1}^{\infty}x^n/n^s$, including Appell's integral, Jonquière's contour integral and many series expansions. Some of the latter are valid for all complex values s except isolated singularities, others for s integral. A table of values for the function is given for 29 values of x between 0 and 1 (mostly near 1) for $s=-\frac{1}{2}, \frac{1}{2}$ and $\frac{3}{2}$. Accuracy to 3 or 4 places is indicated for each entry. The expansion for s a negative integer $-n$ is

$$\phi(x, -n)=(-1)^{n+1}\sum_{p=0}^{n} a_p^n(x-1)^{-p-1},$$

a formula being given for a_p^n in terms of the Stirling numbers. Several difference equations are also given for the a_p^n. Although some of the results are classic, the treatment seems to be novel and is certainly economical. For example, Kummer's formulas for negative integral s are obtained as by-products in deriving the result quoted above.

I. Niven (West Lafayette, Ind.).

Weinbaum, Sidney. On the solution of definite integrals occurring in antenna theory. J. Appl. Phys. 15, 840–841 (1944).

The integral discussed is

$$I=\int_0^h (z^2+r^2)^{-\frac{1}{2}}\exp\{-i\beta(z^2+r^2)^{\frac{1}{2}}\}dz.$$

As a function of $k=-i\beta r$, the integral satisfies a linear differential equation of the second order from which the expansion of I in powers of k is obtained. [The same expansion could have been obtained more readily by expanding the exponential function in the integrand in powers of β and then integrating term by term.]

A. Erdélyi (Edinburgh).

Wheeler, T. S. A note on the evaluation of the Schrödinger hydrogenic intensity integral. Proc. Roy. Irish Acad. Sect. A. 50, 7–12 (1944). [MF 11371]

The author evaluates the general integral

$$\int_0^\infty u^{s+1}\exp[-u(1+x)/2]L_r^s(xu)L_t^{s-2}(u)du,$$

where $L_r^s(x)$ is the associated Laguerre polynomial, by an extension of the method originally used by Schrödinger for special values of the parameters. This method is simpler and more direct than that of Epstein [Proc. Nat. Acad. Sci. U. S. A. 12, 629–633 (1926)] or of Gordon [Ann. Physik (5) 2, 1031–1056 (1929)].

M. C. Gray.

Sinha, S. Infinite integrals involving generalised hypergeometric function. Bull. Calcutta Math. Soc. 36, 15–30 (1944). [MF 11213]

The object of this paper is to evaluate six infinite integrals of which the first is

$$I=\int_0^\infty x^\lambda J_\nu(ax)\,{}_pF_q(-b^2x^2)dx.$$

[The parameters of the generalized hypergeometric series ${}_pF_q$ have been omitted.] In four other integrals, the Bessel function is replaced by a Bessel function of imaginary argument or by a product of Bessel functions; and in one further integral the variable of ${}_pF_q$ is $-b^{-2}x^{-2}$.

The integral I is evaluated by expanding the hypergeometric series and integrating term-by-term. In the author's opinion "this step is easily justifiable when $p-q\leqq 1$." The reviewer cannot see any justification for this procedure which, moreover, leads to infinite integrals like

$$\int_0^\infty x^{\lambda+2n}J_\nu(ax)dx, \qquad n=0, 1, 2, \cdots,$$

of which at least the later ones certainly diverge. The paper seems to abound in divergent infinite integrals, very few of the numerous "results" seem to have any sense at all, and even those that are not void of any meaning do not appear to have been proved.

A. Erdélyi (Edinburgh).

Jackson, F. H. Basic double hyper-geometric functions. II. Quart. J. Math., Oxford Ser. 15, 49–61 (1944). [MF 11801]

[The first part appeared in the same J. 13, 69–82 (1942); these Rev. 4, 141.] Properties are given of ten basic functions defined by means of double power series of the type

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A_{m,n}x^my^nq^{n(n-1)},$$

in which the coefficient $A_{m,n}$ is expressed in terms of basic numbers of type $(a)_p$, where

$$(c)_N=(1-q^c)(1-q^{c+1})\cdots(1-q^{c+N-1}),$$

and ordinary factorials.With the aid of elementary operations relations are obtained between these functions and some functions obtained in the first part. Basic Laguerre polynomials, basic Bessel functions and a basic hypergeometric function of argument $x+y-xy$ are considered. Analogues are obtained of known theorems.

H. Bateman (Pasadena, Calif.).

Bose, B. N. On certain transformations in generalized hyper-geometric series. Bull. Calcutta Math. Soc. 36, 74–79 (1944). [MF 11353]

Three identities (7), (15), (23) are obtained between six infinite series of generalized hypergeometric functions, each identity connecting two such infinite series. The proofs are rather roundabout and the author fails to point out that in each case the infinite series of hypergeometric functions can be summed and that the three quoted equations simply express equality of two different expansions of multiples of, respectively,

$${}_3F_2\left[\begin{matrix}k-\alpha-\beta,\ 1+a-c-d;\ z\\ k-\alpha,\ k-\beta,\ 1+a-d\end{matrix}\right],$$

$${}_2F_2\left[\begin{matrix}k-\alpha-\beta,\ 1+a-c-d;\ z\\ k-\alpha,\ k-\beta\end{matrix}\right],$$

$${}_2F_1\left[\begin{matrix}k_1,\ 1+a-c-d;\ z\\ k-\alpha\end{matrix}\right].$$

This omission is the more surprising as two of the six requisite summations appear as "particular cases" (9), (24) in the paper. All six summations are easily proved by expanding the hypergeometric series and rearranging the double infinite series in powers of z. A considerable number of particular cases are listed.

A. Erdélyi (Edinburgh).

Differential Equations

★Bieberbach, Ludwig. Theorie der Differentialgleichungen. Dover Publications, New York, 1944. xiii +399 pp. $3.50.

Reprint of the third (latest) edition of vol. 6 of the collection Die Grundlehren der Mathematischen Wissenschaften. The original was published by J. Springer, Berlin, 1932.

Dubois-Violette, Pierre-Louis. Sur les points singuliers exceptionnels des équations différentielles du premier ordre, considérés comme limites de points singuliers simples. C. R. Acad. Sci. Paris **217**, 567–569 (1943). [MF 11672]

When $\epsilon\rightarrow 0$ the solution $y=(1-\epsilon/x)^{1/\epsilon}$ of $x(x-\epsilon)y'=y$ tends towards the solution $y=\exp(-1/x)$ of $x^2y'=y$. The object of the present paper is to show that similar results hold between the solutions of

$$x(x-\epsilon)y'=ay+bx+\sum a_{ij}x^iy^j,\qquad a\neq 0,$$

and the solutions of the same equation with $\epsilon=0$ (here $i, j=2, 3, \cdots$). *F. G. Dressel* (Durham, N. C.).

Beckerley, J. G. Expansion of positive energy Coulomb wave functions in powers of the energy. Phys. Rev. (2) **67**, 11–14 (1945). [MF 11875]

The differential equation

$$R_l''+2r^{-1}R_l'+[k^2-2\beta r^{-1}-l(l+1)r^{-2}]R_l=0$$

arising in the quantum mechanical treatment of the motion of charged particles in the field of a charged nucleus is discussed for small values of kr. It is found that the solution of this equation can be expanded in terms of an ascending power series in $(kr)^2$, the first term being $(kr)^l$. The coefficient of each term in the power series is a finite series in the functions $(-2\beta r)^{p/2}J_p[2(-2\beta r)^{\frac{1}{2}}]$, p integral. *H. Feshbach* (Cambridge, Mass.).

Caccioppoli, Renato, e Ghizzetti, Aldo. Ricerche asintotiche per una particolare equazione differenziale non lineare. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 427–440 (1942) = Ist. Naz. Appl. Calcolo (2) no. **129**. [MF 11523]

Let $f(x)$ be a periodic function and $\phi(y')$ be a monotone increasing function. If

$$(*)\qquad \liminf_{|y'|\rightarrow\infty}\phi(y')/y'>0,$$

then the equation $y''+\phi(y')+y=f(x)$ has a unique periodic solution to which all other solutions tend as $x\rightarrow+\infty$. [A stronger theorem with (*) replaced by $\phi(y')\rightarrow\infty$ as $y'\rightarrow\infty$ has been given by the reviewer [J. Math. Phys. Mass. Inst. Tech. **22**, 181–187 (1943); these Rev. **5**, 183].] *N. Levinson* (Cambridge, Mass.).

Caccioppoli, Renato, e Ghizzetti, Aldo. Ricerche asintotiche per una classe di sistemi di equazioni differenziali ordinarie non lineari. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **3**, 493–501 (1942) = Ist. Naz. Appl. Calcolo (2) no. **131**. [MF 11514]

The authors consider the system of differential equations

$$(*)\qquad \ddot{u}_i+\phi_i(\dot{u}_1, \cdots, \dot{u}_n)+\sum_{k=1}^{n}A_{ik}u_k=f_i(t),$$

$i=1, 2, \cdots, n$. Let $A_{ik}=A_{ki}$ be constants for which $\sum A_{ik}\xi_i\xi_k$ is positive definite. Let $f_i(t)$ be continuous and have a common period ω and let $\phi_i(\dot{u}_1, \cdots, \dot{u}_n)$ have continuous first partial derivatives for all values of $\dot{u}_i$. Denote $\partial\phi_i/\partial\dot{u}_k$ by ϕ_{ik}. If $\sum\phi_{ik}\xi_i\xi_k$ is positive definite then either no solution of (*) is stable or (*) has a solution to which all other solutions tend as $t\rightarrow\infty$. (A solution is stable if the point $(u_1(t), \cdots, u_n(t), \dot{u}_1(t), \cdots, \dot{u}_n(t))$ remains bounded in $2n$-dimensional space as $t\rightarrow\infty$.) Extensions are made in certain cases where the quadratic form is positive semidefinite. *N. Levinson* (Cambridge, Mass.).

Koukless, I. S. Sur les conditions nécessaires et suffisantes pour l'existence d'un centre. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 160–163 (1 plate) (1944). [MF 11631]

In order that the origin should be a center for the equation $yy'+x=F(x, y)$, where F is an analytic function not containing linear terms, it is necessary and sufficient that, $\Phi(\alpha, \beta)$ being a given function analytic for small α and β and otherwise arbitrary, there exists a function $V(x, y)$ analytic at the origin with partial derivatives satisfying $V_y(0, 0)=0$, $V_{yy}(0, 0)=1$, and $\partial^{2n}V(0, 0)/\partial x^{2n}=0$ for $n>1$, and also satisfying identically

$$-x+F(x, y)=\{y\Phi(x, V)-V_x(x, y)\}/V_y.$$

N. Levinson (Cambridge, Mass.).

Koukless, I. S. Sur quelques cas de distinction entre un foyer et un centre. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 208–211 (1944). [MF 11636]

The author considers a number of problems related to the differential equation

$$(*)\qquad y'=y^{-1}F(x, y).$$

Problem 1. Suppose that $f_0(x)$, $\phi_0(y)$ and $\phi_1(y)$ and their derivatives are continuous; that these functions, while not necessarily analytic at the origin, are analytic for all small positive and negative arguments; and that $f_0(0)=0$, $f_0'(0)\neq 0$. A fourth function $f_1(x)$ is defined only for $x\geqq 0$ (or for $x\leqq 0$). Find $f_1(x)$ for $x<0$ so that the origin is a center for (*) with $F(x, y)=f_0(x)\phi_0(y)+f_1(x)\phi_1(y)$. The author gives conditions under which the problem can be solved. Problem 2. In (*) $F=\omega_0(x)+y\omega_1(x)+y^2\omega_2(x)$, where two of the three functions ω_i are given. Find the third function so that, if the origin is a focal point, it is a center. Here the third function is found from the equations

$$\omega_2=f(x);\quad \omega_1=p[f(x)\phi(x)-\phi'(x)];$$
$$\omega_0=q\phi(x)[f(x)\phi(x)-\phi'(x)],$$

where f and ϕ are arbitrary functions and p and q arbitrary constants. Other problems are considered. *N. Levinson.*

Koukless, I. S. Sur deux groupes fondamentaux de points singuliers. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 253–255 (1944). [MF 11638]

Let

$$F(x, y)=\sum_{k=n}^{\infty}\lambda_kx^k+y\sum_{k=0}^{\infty}a_kx^k+y^2\Phi(x, y),$$

where Φ is analytic at the origin. The nature of the singularity of the differential equation $y'=-F(x, y)/y$ at the origin is considered. The author proves that for the origin to be a focal point (this includes the case of a center) it is necessary and sufficient that: (a) $n=2k-1$, where k is a

positive integer; (b) all $a_i=0$ for $i<k-1$; (c) $\lambda_{2k-1}>0$; (d) $|a_{k-1}|<2(k\lambda_{2k-1})^{\frac{1}{2}}$. Further results are given.

N. Levinson (Cambridge, Mass.).

Otrokov, N. Sur le nombre des cycles limites au voisinage d'un foyer. C. R. (Doklady) Acad. Sci. URSS (N.S.) 43, 98–101 (1944). [MF 11612]

The author considers the equation

$$(*)\qquad \frac{dy}{dx}=\frac{x+\alpha y+P(x, y)}{\alpha x-y+Q(x, y)},$$

where

$$P(x, y)=\sum_{i+k=2}^{N} b_{ik}x^iy^k,$$

and $Q(x, y)$ is the same as $P(x, y)$ except that b_{ik} is replaced by a_{ik}; N is a fixed integer and α, a_{ik} and b_{ik} are real. The author proves that for any $\delta>0$ there are values of α, a_{ik} and b_{ik} such that (*) has a focal singular point at the origin within δ of which there are $(N^2+2N-4)/2$ limit cycles if N is even and $(N^2+2N-9)/2$ limit cycles if N is odd. The limit cycles all enclose the origin. *N. Levinson.*

Nougmanova, Ch. Sur la stabilité des mouvements périodiques. C. R. (Doklady) Acad. Sci. URSS (N.S.) 42, 202–204 (1944). [MF 11634]

The author considers a system of differential equations and sets up a function of Liapounoff to determine the stability, in the sense of Liapounoff, of periodic solutions. The results hinge on certain finite sequences of determinants being positive. The criteria can be applied directly to a linear homogeneous system of differential equations with periodic coefficients. *N. Levinson.*

Lourie, A. I., and Postnikov, V. N. Concerning the theory of stability of regulating systems. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 8, 246–248 (1944). (Russian. English summary) [MF 11598]

An automatically controlled system, studied by means of a perturbation procedure by Bulgakov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 37, 250–253; these Rev. 4, 275], is studied here by methods of Liapounoff. A change of variables is made and a Liapounoff function is constructed for the system of differential equations. An explicit determination of the range of stability is made.

N. Levinson (Cambridge, Mass.).

Andronow, A., et Mayer, A. Le problème de Mises dans la théorie de la régulation directe et la théorie des transformations ponctuelles des surfaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) 43, 54–58 (1944). [MF 11609]

A new approach is given to the study of the system

$$(*)\qquad \dot{x}=z,\quad \dot{y}=-x,\quad \dot{z}=F(x, y, z),$$

where $F(x, y, z)=-Ax+y+R$; R is $-\frac{1}{2}$ if $z>0$ or if $z=0$ and $-Ax+y>\frac{1}{2}$; R is $\frac{1}{2}$ if $z<0$ or if $z=0$ and $-Ax+y<-\frac{1}{2}$. If $z=0$ and $|-Ax+y|\leqq\frac{1}{2}$, $F=0$. The solutions of (*) emanating from the plane $z=0$ can be studied directly since, once $z\neq0$, (*) is linear with constant coefficients. If (*) carries a point P_0 out of the plane $z=0$ it is shown that this point returns to the plane $z=0$ at a point P_1. A transformation of this plane into itself is defined. A study of this transformation leads to a very neat solution of the problem of determining the stationary solutions of (*) including the determination of stability in the large.

N. Levinson (Cambridge, Mass.).

Amerio, Luigi. Un preliminare teorema di analisi per lo studio dei moti con resistenza passiva. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 3, 415–426 (1942) = Ist. Naz. Appl. Calcolo (2) no. 128. [MF 11508]

Let $P(t)$ be a vector with coordinates $x_1(t), \cdots, x_n(t)$. Consider the system of differential equations

$$(*)\qquad \ddot{P}(t)+\text{grad } U(P)=F(t, P, \dot{P}, \ddot{P}),$$

where the vector $F(t, \alpha, \beta, \gamma)$ is a continuous function for $t\geqq t_0$ and all vectors α, β, γ, and $U(P)$ is a continuous scalar function of position together with its partial derivatives. Let $U(P)\geqq0$. Let the scalar product

$$\beta\cdot F(t, \alpha, \beta, \gamma)\leqq|\beta|\omega(t, |\alpha|, |\beta|),$$

where $\omega(t, x, y)$ is continuous and $\omega(t, x, y)$ and $y\omega(t, x, y)$ are nondecreasing functions of x and y for $x\geqq0$, $y\geqq0$, $t\geqq t_0$. Furthermore, suppose that all the solutions $(z, \dot{z})$ of $\ddot{z}=\omega(t, z, \dot{z})$ which are finite for $t=t_0$ exist for $t_0\leqq t<\infty$. Then the solutions of (*) corresponding to arbitrary initial conditions at $t=t_0$ are prolongable over $t_0\leqq t<\infty$. Interesting applications are made. *N. Levinson.*

Bicadze, A. V. On a general representation of solutions of linear elliptic differential equations. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 613–622 (1943). (Georgian. Russian summary) [MF 11707]

Vekua, Ilja. Remarks on the general representation of solutions of differential equations of elliptic type. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 385–392 (1943). (Georgian. Russian summary) [MF 11704]

Vekua, Ilja. On some fundamental properties of metaharmonic functions. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 281–288 (1943). (Russian. Georgian summary) [MF 11701]

Fundamental properties of metaharmonic functions, that is, regular solutions $U(P)\equiv U(x_1, \cdots, x_p)$ of the partial differential equation $\Delta U+\lambda^2U=0$, where λ is a constant and Δ is the Laplace operator, are discussed. It is shown that, if $U(P)$ is metaharmonic in a finite domain T whose boundary S consists of a finite number of mutually exclusive regular hypersurfaces, and if $U(P)$ and its first order partial derivatives are continuous in $T+S$, then $U(P)$, for P in T, can be given in terms of an integral over S. It follows that every metaharmonic function is analytic in its domain of definition. For infinite regions T, the integral representation of $U(P)$ holds only if additional restrictions are satisfied at infinity. A series expansion for functions metaharmonic in an annular domain is given. Solutions of Dirichlet and Neumann problems for functions metaharmonic in infinite domains are presented.

E. F. Beckenbach (Austin, Tex.).

Vecoua, Elias. On metaharmonic functions. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 12, 105–174 (1943). (Russian. Georgian and English summaries) [MF 11688]

The results of the paper reviewed above are given in much greater detail, in some instances with different proofs. In addition, there is a discussion of n-metaharmonic func-

tions, that is, regular solutions of the equation

$$(1)\qquad \Delta^n U + a_1\Delta^{n-1}U + \cdots + a_n U = 0,$$

where the a_i are constants. In particular, it is shown that all regular solutions of equation (1) are of the form

$$U(P) = \sum_{i=1}^{m}\sum_{j=0}^{k_i-1} r^j\, \partial^j U_{ij}(P;\kappa_i)/\partial r^j + V(P),$$

where r is the distance of P from the origin; κ_i are the distinct roots of the equation

$$(2)\qquad x^n - a_1 x^{n-1} + \cdots + (-1)^n a_n = 0$$

which are different from zero; k_i is the multiplicity of the root κ_i; the $U_{ij}(P;\kappa_i)$ are solutions of the equations $\Delta U + \kappa_i U = 0$, respectively; and $V(P)$ is a k-harmonic function, where k is the multiplicity of zero as a root of equation (2). *E. F. Beckenbach* (Austin, Tex.).

Jackson, Dunham. The harmonic boundary value problem for an ellipse or an ellipsoid. Amer. Math. Monthly **51**, 555–563 (1944). [MF 11731]

Consider the ellipse $x = a\cos\theta$, $y = b\sin\theta$, $0 \leqq \theta \leqq 2\pi$, $a > b$, and let

$$f(\theta) = a_0/2 + \sum (a_n \cos n\theta + b_n \sin n\theta)$$

be the assigned boundary values of a function harmonic within the ellipse. Write this harmonic function in the form

$$F(x,y) = a_0/2 + \sum (a_n U_n(x,y) + b_n V_n(x,y)),$$

where U_n and V_n are linear combinations of harmonic polynomials in x and y which reduce, respectively, to $\cos n\theta$ and $\sin n\theta$ on the ellipse. The present paper gives methods for determining the functions U_n and V_n. A similar problem for an arbitrary ellipsoid is partially carried through. *F. G. Dressel* (Durham, N. C.).

Keldych, M. Sur le problème de Dirichlet. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 308–309 (1941). [MF 10987]

Various procedures for determining a harmonic function with given continuous boundary values will yield the solution of the classical Dirichlet problem if it exists. If such a solution does not exist, all these methods yield the same harmonic function which is called the solution of the generalized Dirichlet problem. It can be considered as a functional continuation of the solution of the classical Dirichlet problem. This continuation satisfies the following conditions. (1) $A_f(P)$ is an operator which associates with every continuous function on the boundary C of the domain D a function which is harmonic in D. (2) If for some $f(P)$, $P\epsilon C$, the Dirichlet problem has a solution, $A_f(P)$ yields this solution. (3) A_f is a distributive operator. (4) $A_f(P)$ satisfies the inequalities $\min f(P) \leqq A_f(P) \leqq \max f(P)$. In the present note the author shows that the above conditions determine uniquely the functional continuation of the classical Dirichlet problem. *S. Bergman.*

Ghizzetti, Aldo. Sui problemi di Dirichlet per la striscia e per lo strato. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (7) **13**, 617–649 (1942) = Ist. Naz. Appl. Calcolo (2) no. **125**. [MF 11521]

The solution of Laplace's equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ for the infinite strip $-\infty < x < \infty$, $0 < y < a$ with the boundary conditions $u(x,0) = f(x)$, $u(x,a) = g(x)$ ($f(x)$ and $g(x)$ prescribed functions of x) is given by the Fourier integral representation

$$(*)\quad U(x,y) = \frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^{\infty}\left[f(s)\frac{\sinh\lambda(a-y)}{\sinh\lambda a} + g(s)\frac{\sinh\lambda y}{\sinh\lambda a}\right]\cos\lambda(s-x)\,ds.$$

The author discusses this integral in order to show under what hypotheses on $f(x)$ and $g(x)$ (*) satisfies Laplace's equation and the prescribed boundary conditions. The analogous problem for three dimensions is also discussed. *A. E. Heins* (Cambridge, Mass.).

Fedoroff, W. S. Sur les fonctions harmoniques conjuguées dans l'espace. Rec. Math. [Mat. Sbornik] N.S. **13**(55), 287–300 (1943). (Russian. French summary) [MF 11654]

Let $\varphi(x,y,z)$ and $\psi(x,y,z)$ be harmonic and real-valued, defined in a three-dimensional domain D. (a) Under what conditions will any analytic function $f(\zeta)$ be a (complex-valued) harmonic function of x, y, z if we set $\zeta = \varphi + i\psi$? (b) Under what conditions will any harmonic function $f(p,q)$ be a harmonic function of x, y, z if we set $p = \varphi$, $q = \psi$? The necessary and sufficient condition in both cases is the same, namely, (*) $|\nabla\varphi| = |\nabla\psi|$, $\nabla\varphi\cdot\nabla\psi = 0$ at every point of D. However, as the author points out, even the function $\varphi = (x^2+y^2+z^2)^{-\frac{1}{2}}$ has no corresponding ψ satisfying (*). If $\varphi = \varphi(x,y)$ is a harmonic function of x, y, it must be of a very special structure in order that there should exist a $\psi(x,y,z)$ satisfying (*), and the author gives a complete answer in this case. *A. Zygmund.*

Lozinski, S. On subharmonic functions and their application to the theory of surfaces. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 175–194 (1944). (Russian. English summary) [MF 11530]

Using functions of class PL (functions having subharmonic logarithms) and averaging functions, the author obtains the isoperimetric characterization [E. F. Beckenbach and T. Radó, Trans. Amer. Math. Soc. **35**, 662–674 (1933)] of surfaces of nonpositive Gaussian curvature under somewhat less restrictive differentiability conditions. It is shown that, if S is a surface of class C'' given in isothermic representation for parameters (u, v) in a domain D, if the Gaussian curvature K satisfies $K \leqq 0$ wherever K is defined on S, if Γ is a rectifiable curve lying in D, and if l is the length of the image Γ^* of Γ on S and a the area of the part S^* of S enclosed by Γ^*, then l and a satisfy the isoperimetric inequality (1) $a \leqq l^2/(4\pi)$.

We note that Shiffman [Bull. Amer. Math. Soc. **49**, 857 (1943)] has also announced a lessening of differentiability conditions, showing first by purely elementary methods that (1) holds on polyhedral surfaces for which the sum of face angles is not less than 2π at each interior vertex, and then employing a limiting process for surfaces of nonpositive curvature. Several other generalizations of results concerning functions of class PL and surfaces of nonpositive curvature are given, one of which we note has been anticipated by A. F. Monna [Nederl. Akad. Wetensch., Proc. **45**, 687–689 (1942); these Rev. **5**, 241].

It is shown further that, if the function $w = f(z)$ maps the circle $|z| < 1$ conformally on the interior of a rectifiable plane curve Γ, and if Γ^* is the image of a circle tangent to $|z| = 1$ from within, then the length and the total rotation

of Γ^* are less than or equal to the length and the total rotation, respectively, of Γ. *E. F. Beckenbach.*

Grioli, G. Funzioni di Green per le piastre elastiche sottili. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **3**, 293–316 (1942) = Ist. Naz. Appl. Calcolo (2) no. **144**. [MF 11727]

The author studies some Green's functions which arise in the theory of deflection of elastic plates under various boundary conditions. The convergence of the series expansions thus obtained is studied and some numerical results are given. *A. E. Heins* (Cambridge, Mass.).

Taylor, James H. On the determination of magnetic vertical intensity, Z, by means of surface integrals. Terr. Magnetism **49**, 223–237 (1944). [MF 11547]

By means of Poisson's integrals, expressions are obtained for the normal derivatives Z_e, Z_i of potentials arising, respectively, from sources external and internal to the sphere. The relation

$$Z_e(f, p)+Z_e(f, p)=-f(p)/a$$

connecting these derivatives with the surface value $f(p)$ of the potentials is a consequence.

In the main problem the given surface value $h(\theta, \phi)$ of a potential function is regarded as the sum of the surface value $f(\theta, \phi)$ of a potential $V_e(f, p)$ arising from sources outside the sphere and of the surface value $g(\theta, \phi)$ of a function V_i arising from sources inside the sphere. With the aid of Poisson integral expressions for V_e and V_i a relation

$$Z_e(f, p)-Z_e(g, p)=Z_e(h, p)+g(p)/a$$

is obtained and is coupled with a relation $Z_e(f, p)+Z_i(g, p)=Z(f, g, p)$ defining $Z(f, g, p)$, which may be regarded as the observed normal force component. These two relations give $Z_e(f, p)$ and $Z_i(g, p)$ in terms of a single function $g(p)$ which is expressed in terms of $h(p)$ and an integral involving $h(Q)$ and $Z(f, g, Q)$. Certain special cases are treated, and a test is made of the proposed method of computation.

H. Bateman (Pasadena, Calif.).

Vernotte, Pierre. Extension, aux milieux illimités, de la méthode générale d'intégration de Fourier, relative aux milieux limités. C. R. Acad. Sci. Paris **217**, 441–443 (1943). [MF 11665]

In a former note [same C. R. **217**, 364–366 (1943); these Rev. **6**, 67] the author developed a specific method of extending a function defined on a finite interval to a function defined on the whole real axis. In the present note he treats a number of instances in which this method is applied to the solution of heat conduction problems in an infinite medium. *E. H. Rothe* (Ann Arbor, Mich.).

Lowan, Arnold N. Note on the problem of heat conduction in a semi-infinite hollow cylinder. Quart. Appl. Math. **2**, 348–350 (1945). [MF 11777]

The following boundary value problem for the hollow cylinder $a<r<b$, $t>0$ is solved by means of a Laplace transformation:

$$K\left(\frac{\partial}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)T(r, z, t)=\frac{\partial}{\partial t}T,$$

$$T(r, 0, t)=T(a, z, t)=T(r, z, 0)=0;\ T(b, z, t)=\varphi(z, t).$$

A solution is also given of the problem where the boundary condition $T(b, z, t)=\varphi(z, t)$ is replaced by $(\alpha\partial/\partial r+\beta)T=0$ for $r=b$. *F. G. Dressel* (Durham, N. C.).

Amerio, Luigi. Sull'applicazione della trasformata di Laplace all'integrazione di equazioni a derivate parziali senza alcun vincolo sul comportamento all'infinito della soluzione. Rend. Circ. Mat. Palermo **63**, 21 pp. (1942) = Ist. Naz. Appl. Calcolo (2) no. **135**. [MF 11526]

With the aid of the double Laplace transform taken over a finite region, the author solves the Cauchy problem for the linear partial differential equation

$$F_{xy}+aF_x+bF_y+CF=H(x, y).$$

Here $H(x, y)$ is a prescribed function of x and y, the a, b, c are constants and F and its two first partial derivatives are prescribed along a given curve. *A. E. Heins.*

Smith, J. J. The extension of the Heaviside expansion theorem to the equations of engineering and physics in curvilinear orthogonal coordinates. J. Franklin Inst. **238**, 245–272 (1944). [MF 11205]

This paper gives a semi-operational derivation of Green's function for three-dimensional wave or heat equations. The general idea may be sketched in the following equivalent formulation. Let (1) $g_{ii}=\lambda_i$, $g_{ij}=0$; $i, j=1, 2, 3$. The boundary conditions are of the special form (2, i) $\partial v/\partial x^i+(-1)^\sigma h_i^\sigma\lambda_i=0$ at $x^i=x_\sigma^i$, $\sigma=1, 2$. The equation solved is written (3) $\nabla^2 v=\rho\delta$, where δ is the Dirac function $\delta(r-r_0)$. Then, with

$$\alpha^2=\sum_{l=2}^{3}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^l}\sqrt{g}\,g^{kl}\frac{\partial}{\partial x^k}, \tag{4}$$

one gets

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x'}\sqrt{g}\,g''\frac{\partial}{\partial x'}v+\alpha^2 v=\rho\delta. \tag{5}$$

The solution of (5) subject to (2, 1) is essentially the standard Green's function argument and not really operational. [A completely operational formulation requires ideas such as those in Bourgin and Duffin, Bull. Amer. Math. Soc. **45**, 859–869 (1939); these Rev. **1**, 180.] The end result is (6) $v=Q(x_1, \alpha)I$, where I is the symbolic integral

$$\int_{x_0'-}^{x_0'+}\lambda_1\rho\delta dx'.$$

Assuming the poles of Q enter only as pairs α_p, $-\alpha_p$, the partial fraction representative of $Q(x, \alpha)$ is

$$\sum_p R_p(x)(1/(\alpha^2-\alpha_p^2))I.$$

Let

$$(1/(\alpha^2-\alpha_p^2))I=G_p;$$

then $(\alpha^2-\alpha_p^2)G_p=I$, where α^2 is the operator in (4). This equation is treated in a way similar to that for (5), etc. Thus one replaces the three-dimensional problem by a succession of one-dimensional problems, noting that, although I is symbolic, the final $\iint I dx^2 dx^3$ can be interpreted by the Dirac function convention, and the end result is a series for the Green's function. *D. G. Bourgin* (Urbana, Ill.).

Green, George. Solutions of problems relating to media in contact by the method of wave-trains. Philos. Mag. (7) **35**, 519–531 (1944). [MF 11342]

The writer considers a rod of material 1 from 0 to a and material 2 from a to b. In essence the method is the following. Write $K(x, t; x_1, k)$ for the disturbance at x and t occasioned by a unit harmonic source at x_1 of frequency k. Of course, K satisfies the differential equations of the problem (either wave or heat conduction) and the boundary conditions. If $f(x)$ is the initial distribution at $t=0$, a com-

bination of K terms is determined that gives the same effect as the instantaneous source of strength $f(x_1)$ (at $t=0$). Then integrating over x_1 from 0 to b and over k from 0 to ∞ yields the final solution. This mode of presentation makes it fairly clear that in essence the method is a combination of the Fourier integral decomposition of a disturbance into harmonic waves and the method of Green's function, noting that we have here a three point ($x=0$, a, b) boundary condition system. *D. G. Bourgin.*

Powsner, A. Sur les équations du type de Sturm-Liouville et les fonctions "positives." C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 367–371 (1944). [MF 11621]

The solution of the boundary value problem

$$\partial^2u/\partial x^2-\partial^2u/\partial y^2-c(p(x)-p(y))u=0,$$

$p(x)=p(-x)$, $u(x,\ 0)=f(x)$, $u_y(x,\ 0)=0$, $f(x)=f(-x)$, $-\infty<x<\infty$, is written in the form

$$T_x{}^yf(x)=u(x,y)=\tfrac{1}{2}[f(x+y)+f(x-y)]-\int_0^\infty f(t)\omega(t,x,y)dt.$$

Conditions are given for the operator $T_x{}^y$ to satisfy the inequality

$$\sup_{-\infty<x,y<\infty}|T_x{}^yf(x)|\leqq C\sup_{-\infty<x<\infty}|f(x)|. \tag{1}$$

Let K be the space of even functions summable on the x-axis, define multiplication in K by

$$f*g=\int_{-\infty}^\infty f(x-t)g(t)dt-\int_{-\infty}^\infty\int_{-\infty}^\infty \omega(x,\xi,\eta)f(\xi)g(\eta)d\xi d\eta,$$

and introduce the norm $\|f\|=\int_{-\infty}^\infty|f(t)|dt$. The author states that K is then a normed ring without unit element in the sense of I. Gelfand [Rec. Math. [Mat. Sbornik] N.S. **9**(51), 3–24 (1941); these Rev. **3**, 51] if and only if condition (1) is satisfied. By introducing a unit element e [in a manner not indicated] a normed ring $\bar{K}$ with a unit element is obtained. A maximal ideal is defined in $\bar{K}$ by each function $\varphi(x,\lambda)$ satisfying $d^2\varphi/dx^2+(\lambda^2-cp(x))=0$; $\varphi(0,\lambda)=1$, $\varphi'(0,\lambda)=0$; that is, the set of elements $\alpha e+f\epsilon\bar{K}(f\epsilon K)$ for which

$$\alpha+\int_{-\infty}^\infty f(t)\varphi(t,\lambda)dt=0.$$

The function $\varphi(x,\lambda)$ is also used to generalize Bochner's theorem on positive functions. A function is defined as L-positive if it can be represented in the form $f(x)=\int_0^\infty\varphi(x,\lambda)d\alpha(\lambda)$, where $\alpha(\lambda)$ is a nondecreasing function of bounded variation. Necessary and sufficient conditions for $f(x)$ to be L-positive are given. *F. G. Dressel.*

Mindlin, J. A. Propagation of waves on the surface of an infinitely long circular cylinder conceived as a cut-out portion of an infinitely elastic space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **42**, 151–155 (1944). [MF 11629]

[In the original the name was misprinted as Minldlin.] The author considers perturbations which do not penetrate far into the medium and so uses functions which are proportional to $K_0(w_1)$, $K_1(w_2)$, respectively, where w_i is of the form $r(\gamma^2-\delta_i^2)^{\frac{1}{2}}$, r being the distance from the axis of the cylinder, δ_1 being p/a, δ_2 being p/b. The quantities γ and p enter into the time factor $\exp i(pt-\gamma z)$, while a, b are the velocities of propagation of longitudinal and transverse waves, respectively.

Much of the analysis relates to the behavior of the function

$$f(x)=K_0(1/x)/K_1(1/x).$$

It is shown in fact that, for $x>0$, $f'(x)<0$, $\lim_{x\to0}f'(x)=-\frac{1}{2}$, $\lim_{x\to0}f(x)=1$, $\lim_{x\to\infty}f(x)=0$, and that the function

$$\Phi(\theta,\xi)=(2\theta^2-b^{-2})^2f(\xi/h_1)-4\theta^2h_1k_1f(\xi/k_1)-2\xi h_1b^{-2}$$

varies monotonically when ξ takes nonnegative values and θ has an arbitrary fixed value exceeding $1/b$. The quantities h_1, k_1 are defined by the equations $h_1=(\theta^2-a^{-2})^{\frac{1}{2}}$, $k_1=(\theta^2-b^{-2})^{\frac{1}{2}}$, $h_1>k_1$. *H. Bateman* (Pasadena, Calif.).

Integral Equations

Azevedo do Amaral, Ignacio. On the solution of integral equations of the first kind. Anais Acad. Brasil. Ci. **16**, 23–39 (1944). (Portuguese) [MF 10889]

The author reconsiders certain relations for integral equations of the first kind that he obtained previously [same Anais **13**, 305–317 (1941); **14**, 87–97, 293–303 (1942); these Rev. **3**, 243; **4**, 99, 279]. As in the previous papers, however, he does not give conditions under which his formulas are valid, and it is not clear what classes of integral equations may be solved by his special methods.

W. T. Reid (Evanston, Ill.).

Sauer, Ludwig. Parametrixmethode zur Lösung von Randwertproblemen. Math. Ann. **118**, 385–440 (1942). [MF 8937]

This is the first of a series of papers on the Hilbert method of the parametrix for the solution of elliptic partial differential equations. It deals with integral equations of the second kind

$$W-\int K\cdot W'dt'=M \tag{1}$$

and with various integral estimates which will be of use in later papers. The author obtains refinements of the conditions which suffice for the validity of the classical theory of integral equations. These conditions are of two kinds, associated with the names of Fredholm and Levi, respectively. In the first type, besides very general assumptions concerning the behavior of the kernel K and of the functions M and W, the assumption is made that there exists an integer m such that the classical theory holds for the nth iterated kernel for all $n\geqq m$. The conclusion is then that the classical theory holds for the original integral equation. In the second type, the author makes essentially the same assumptions as were made by Levi and extends his results somewhat, using the method of splitting of the kernel. *M. Shiffman.*

Michlin, S. Sur un théorème de F. Noether. C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 139–141 (1944). [MF 11613]

With the aid of the theory of linear operators in spaces L_2, the following result of F. Noether is proved in a much simpler way than heretofore. The singular integral equation

$$a(x)\phi(x)-b(x)\int_{-\pi}^{\pi}\phi(s)\cot\tfrac{1}{2}(s-x)ds+\int_{-\pi}^{\pi}K(x,s)\phi(s)ds=f(x),$$

where $K(x,s)$ is regular and $a(x)$ and $b(x)$ satisfy Lipschitz conditions, has a solution if and only if f is orthogonal to all the solutions of the corresponding conjugate homogeneous equation. *W. J. Trjitzinsky* (Urbana, Ill.).

Parodi, Maurice. Sur une propriété d'équations intégrales et intégro-différentielles du type de Volterra. C. R. Acad. Sci. Paris **217**, 523–525 (1943). [MF 11670]

The author shows that, if one knows the solution of the integral equation

$$\lambda f_1(x)+\int_0^x f_1(y)K(x-y)dy=1,$$

the solution of the integral equation

$$\lambda f(x)+\int_0^x f(y)K(x-y)dy=g(x)$$

($g(x)$ prescribed) can be derived from it. This is accomplished with the aid of the unilateral Laplace transform. A similar result holds for integro-differential equations of the type

$$\sum a_n f^{(n)}(x)+\int_0^x f(y)K(x-y)dy=g(x)$$

(a_n's constant). *A. E. Heins* (Cambridge, Mass.).

Ingram, W. H. On the integral equations of continuous dynamical systems. Proc. Nat. Acad. Sci. U. S. A. **30**, 370–376 (1944). [MF 11384]

The author discusses a method for determining the characteristic numbers and functions of a vector integral equation which arises from one-dimensional dynamical systems. [See W. H. Ingram, Philos. Mag. (7) **30**, 16–38 (1940); these Rev. **2**, 99.] *A. E. Heins* (Cambridge, Mass.).

Legras, Paul. Über das asymptotische Verhalten der Erneuerungsfunktion. Mitt. Verein. Schweiz. Versich.-Math. **42**, 183–204 (1942). [MF 11455]

Following Feller [Ann. Math. Statistics **12**, 243–267 (1941); these Rev. **3**, 151], the author investigates the asymptotic behavior of the solution $\phi(t)$ of the integral equation of renewal theory in the special form

$$\phi(t)=p(t)+\int_0^t p(t-x)\phi(x)dx.$$

Imposing stronger conditions on $p(t)$, he also obtains stronger estimates of the deviation from the limiting value.

W. Feller (Providence, R. I.).

Tarján, Rudolf. Untersuchungen zum Erneuerungsproblem nichtkonstanter Gesamtheiten. Mitt. Verein. Schweiz. Versich.-Math. **44**, 95–105 (1944). [MF 11459]

A few simple remarks concerning various representations of the solutions of the integral equation of renewal theory [cf. the preceding review]. The paper contains no novelty from a mathematical point of view. *W. Feller.*

Magnaradze, L. G. On a system of linear singular integro-differential equations and on a linear boundary value problem of Riemann. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **4**, 3–9 (1943). (Russian. Georgian summary) [MF 11694]

A system of singular integro-differential equations of the following type is considered:

$$\Phi(t_0)A(t_0)+\dot\Phi(t_0)B(t_0)-\frac{1}{\pi}\int_C\frac{\Phi(t)P(t_0,t)+\dot\Phi(t)Q(t_0,t)}{t-t_0}dt=F(t_0),$$

where $A(t_0)$, $B(t_0)$, $F(t_0)$, $P(t_0,t)$, $Q(t_0,t)$ are given matrices, all of the same order, whose elements are "sufficiently regular" functions; $\Phi(t)$ is the unknown matrix, C a simple closed curve of "sufficient regularity." The author states that the results can easily be extended to the case where C is the sum of a finite number of simple closed or finite open arcs.

It is pointed out that a certain boundary value problem in the theory of functions of a complex variable, known as the problem of Riemann, can be reduced to the solution of a system of integro-differential equations of the above type. The object of the paper is to reduce the given system of integro-differential equations to an equivalent system of integral equations. This is done in several ways. The different methods correspond to the different conditions imposed on the given matrices. These conditions involve the existence (everywhere on C) of the inverses of some of the given matrices and certain others derived from them. The author states that the details of the work as well as the study of the existence of solutions of the reduced system of integral equations are reserved for a future paper.

H. P. Thielman (Ames, Iowa).

Magnaradze, L. G. Theory of a class of linear singular integro-differential equations and its applications to the problem of vibrations of an airfoil of finite span, the collision with the surface of water, and analogous problems. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **4**, 103–110 (1943). (Russian. Georgian summary) [MF 11696]

The following linear singular integro-differential equation is considered:

$$\sum_{r=0}^{m} a_r(t_0)\varphi^{(r)}(t_0)-\sum(1/\pi)\int_C K_r(t_0,t)\varphi^{(r)}(t)dt/(t-t_0)=f(t_0),$$

where $a_r(t_0)$, $f(t_0)$, $K_r(t_0,t)$ are given functions, sufficiently often differentiable, and C is a simple closed curve. The author states that the results can easily be generalized to the case where C consists of the sum of a finite number of closed or finite open arcs.

The connection between the given equation and several applied problems is pointed out. The object of the paper is to reduce the given equation under various restrictions to equivalent integral equations of the Fredholm type. Under certain conditions the reduced integral equation is regular, while in other cases it is singular. Only an outline of the procedure is given. The author states that the details are reserved for a future paper. *H. P. Thielman.*

Dolph, C. L. Non-linear integral equations of the Hammerstein type. Proc. Nat. Acad. Sci. U.S.A. **31**, 60–65 (1945). [MF 11783]

The object of this paper is to prove the existence of a solution $\psi(x)$ of the integral equation

$$\psi(x)=\int_a^b K(x,y)f\{y,\psi(y)\}dy$$

under more general conditions on $f(x,y)$ than those considered by Hammerstein [Acta Math. **54**, 117–176 (1930)] and Iglisch [Math. Ann. **101**, 98–119 (1929)]. It is stated that, under natural generalizations of some of the stronger conditions given in the above mentioned papers, the corresponding earlier proofs can readily be generalized. In contrast to this it is pointed out that a more general case requires an entirely different method of proof. The result

for this case is stated in the form of a theorem with a sketch of its proof. The author states that complete proofs of the results announced in this paper will be published later.

H. P. Thielman (Ames, Iowa).

Lotkin, Mark. On a certain type of nonlinear integral equations. Bull. Amer. Math. Soc. **50**, 833–841 (1944). [MF 11552]

The paper considers an integral equation of the form

$$\varphi(x)=\lambda[f(x)+\sum_{i=1}^{m} G_i(x,\varphi)]$$

with

$$G_i(x,\varphi)=\int_a^b\cdots\int_a^b K_i(x,s_1,\cdots,s_i)\times F_i(s_1,\cdots,s_i,\varphi(s_1),\cdots,\varphi(s_i))ds_1\cdots ds_i,$$

with f and φ of L^2, and proves the existence of an eigenvalue (presumably a value of λ for which the equation has a solution). The functionals $G_i(x,\varphi)$ are assumed to be completely continuous, that is, $\lim_n \int w\varphi_n=\int w\varphi$ for all w of L^2 implies

$$\lim_n \int_a^b [G_i(x,\varphi_n)-G_i(x,\varphi)]^2dx=0.$$

The method of procedure is to introduce a complete orthonormal system of functions $\{w_n(x)\}$ and reduce the problem to the convergence of a sequence of functions minimizing the functional

$$J(v_n)=2\left[\int fv_n+\sum_{i=1}^{m} e_i\int G_i(x,v_n)v_ndx\right]$$

with $v_n(x)=\sum_{k=1}^{n}c_{nk}w_k(x)$. In order to effect equivalence with the integral equations problem the K_i, F_i and e_i are assumed to satisfy a convenient system of linear integral equations. Application is made to the case where $K_i(x,s_1,\cdots,s_i)$ are continuous in x, and $F_i(\varphi)=\varphi(s_1)\cdots\varphi(s_i)$. We might note that the final statement that

$$\varphi(x)=\lambda\int K_m(x,s_1,\cdots,s_m)\varphi(s_1)\cdots\varphi(s_m)ds_1\cdots ds_m$$

has at least one finite eigenvalue is obviously not always true for $m=1$ unless the trivial solution $\varphi(x)=0$ be admitted.

T. H. Hildebrandt (Ann Arbor, Mich.).

Trjitzinsky, W. J. Singular non-linear integral equations. Duke Math. J. **11**, 517–564 (1944). [MF 11088]

The paper is concerned with non-linear integral equations of the two types

$$\varphi(x)+\int_0^1 K(x,t)f(t,\varphi(t))dt=0, \tag{1.1}$$

$$\varphi(x)+\int_0^1\left[\int_0^1 K(x,t)h(t,y,\varphi(y))dt\right]dy=0, \tag{1.2}$$

which are singular in the sense that the integral

$$\int_0^1\int_0^1 K^2(x,t)dxdt$$

may not exist. It is assumed that the integrals

$$K_1^2(x)=\int_0^1 K^2(x,t)dt,\quad K_2^2(t)=\int_0^1 K^2(x,t)dx$$

do exist, and the main body of the discussion is concerned with proving the existence of solutions under various sets of hypotheses. As typical results the following may be quoted. (I) If

(i) $\displaystyle\int_0^1 K(x,t)f(t,0)dt\subset L_2,$

(ii) $|f(t,u')-f(t,u)|<\mu|u'-u|/K_1(t),$

(iii) $\displaystyle\rho=\int_0^1 (K_2^2(t)/K_1^2(t))dt$

exists and (iv) $\mu^2\rho<1$, the equation (1.1) has a solution (unique) in L_2. (II) If

(i) $\displaystyle\int_0^1\left[\int_0^1 K(x,t)h(t,y,0)dt\right]dy\subset L_2,$

(ii) $|h(x,y,u')-h(x,y,u)|<\mu|u'-u|/K_1(x),\ 0\leqq x,y\leqq 1,$

(iii) $\displaystyle\rho(x)=\int_0^1 (K^2(x,t)/K_1^2(t))dt$

exists, (iv) $\rho=\int_0^1\rho(x)dx$ exists and (v) $\mu^2\rho<1$, the equation (1.2) has a solution (unique) in L_2. The method is mainly that of successive approximations.

R. E. Langer.

Theory of Probability

Kaplansky, Irving. Symbolic solution of certain problems in permutations. Bull. Amer. Math. Soc. **50**, 906–914 (1944). [MF 11562]

The author applies Poincaré's formula

$$P_0=1-\sum p(A_i)+\sum p(A_iA_j)+\cdots,$$

where $p(A_i\cdots A_k)$ denotes the probability of the joint occurrence of the events $A_i\cdots A_k$, and P_0 is the probability that none of $A_1,\cdots,A_n$ occurs. He assumes further that $p(A_{i_1}\cdots A_{i_k})$ is either zero or a function ϕ_k of k alone (quasi-symmetry). Exact formulas are found for certain card-matching problems as well as for certain more complicated problems. In the former case approximate formulas also are found.

L. Carlitz (Durham, N. C.).

Ott, E. R. Difference equations in average value problems. Amer. Math. Monthly **51**, 570–578 (1944). [MF 11734]

The author applies linear difference equations to the solution of examples in elementary probability.

E. Lukacs (Berea, Ky.).

Dieulefait, C. The multidimensional Gaussian distribution and its generalization. An. Soc. Ci. Argentina **136**, 193–215 (1943). (Spanish) [MF 11917]

An exposition of the multidimensional Gaussian distribution with particular emphasis on Hermite polynomials.

W. Feller (Providence, R. I.).

Geary, R. C. Extension of a theorem by Harald Cramér on the frequency distribution of the quotient of two variables. J. Roy. Statist. Soc. (N.S.) **107**, 56–57 (1944). [MF 11962]

The theorem mentioned in the title [Cramér, Random Variables and Probability Distributions, Cambridge University Press, 1937] is generalized to the case of dependent random variables. According to the author, "it is doubtful if the present theorem has many practical applications."

W. Feller (Providence, R. I.).

Birnbaum, Z. W., and Zuckerman, Herbert S. An inequality due to H. Hornich. Ann. Math. Statistics **15**, 328–329 (1944). [MF 11330]

Let $x_1, \cdots, x_n$ be independent symmetric chance variables with identical distributions. The authors prove the following inequality, due to Hornich:

$$D_n \geqq \frac{D_1 n}{2^{n-1}}\binom{n-1}{[n/2]},$$

where $D_i = E|x_1 + \cdots + x_i|$. If the hypothesis of symmetry is replaced by the hypothesis $E(x_i) = 0$,

$$D_n \geqq \frac{D_1 n}{2^n}\binom{n-1}{[n/2]}.$$

D. Blackwell (Washington, D. C.).

Hotelling, Harold. Note on a matric theorem of A. T. Craig. Ann. Math. Statistics **15**, 427–429 (1944). [MF 11761]

A theorem of A. T. Craig [Ann. Math. Statistics **14**, 195–197 (1943); these Rev. **5**, 127] is as follows. If A and B are the symmetric matrices of two homogeneous quadratic forms in n variates, which are normally and independently distributed with zero means and unit variances, a necessary and sufficient condition for the independence in probability of these two forms is that $AB = 0$. The present author points out that the proof given for the necessity of the conditions was inadequate and supplies a detailed proof covering this point. *C. C. Craig.*

Dubrovsky, V. Investigation of purely discontinuous random processes by means of integro-differential equations. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **8**, 107–128 (1944). (Russian. English summary) [MF 11462]

The notion of Markov processes (definite stochastic processes) and their reduction to differential and integral equations is due to Kolmogoroff [Math. Ann. **104**, 415–458 (1931)]. The necessary existence and uniqueness theorems have been provided by Feller [Math. Ann. **113**, 113–160 (1936)]. Both authors considered the case where the real axis is the label space. The author studies the "purely discontinuous processes" [Feller, l. c.; also Trans. Amer. Math. Soc. **48**, 488–515 (1940); these Rev. **2**, 101] in an arbitrary abstract space. [A preliminary announcement appeared in C. R. (Doklady) Acad. Sci. URSS (N.S.) **19**, 439–446 (1938). Cf. also Doeblin, Skand. Aktuarietidskr. **1939**, 211–222; these Rev. **1**, 247.] *W. Feller.*

Gnedenko, B. V. On the growth of homogeneous random processes with independent single-type increments. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 90–93 (1943). [MF 11170]

The process considered is a one parameter family of random variables $\xi(t)$ such that (1) the distribution function of $\xi(t_2) - \xi(t_1)$ depends only on $|t_2 - t_1|$; (2) the increments of $\xi(t)$ over nonoverlapping intervals are statistically independent; (3) for all t the distribution functions of $\xi(t_0+t) - \xi(t_0)$ belong to the same type. It follows from a known theorem of Khintchine and Lévy that constants α $(0 \leqq \alpha \leqq 2)$, β $(|\beta| \leqq 1)$, γ and $c(c \geqq 0)$ exist such that the distribution function $F(x, t)$ of $\xi(t_0+t) - \xi(t_0)$ satisfies the equation

$$F(x, t) = F(xt^{-1/\alpha} - a(t), 1),$$

where

$$a(t) = \begin{cases} \gamma(t - t^{1/\alpha}), & \alpha \neq 1, \\ 2\pi^{-1} c\beta t \log t, & \alpha = 1. \end{cases}$$

The author proves three theorems about processes with $0 < \alpha < 2$. The character of the results is exemplified by the following theorem. If $u(t)$ is nondecreasing, then, with probability 1, the limit superior (as $t \to \infty$) of

$$|\xi(t - t_0) - \xi(t_0) - a(t)|/u(t)$$

is either zero or infinity. *M. Kac* (Ithaca, N. Y.).

Cameron, R. H., and Martin, W. T. Evaluation of various Wiener integrals by use of certain Sturm-Liouville differential equations. Bull. Amer. Math. Soc. **51**, 73–90 (1945). [MF 11820]

The authors continue their studies [cf. Ann. of Math. (2) **45**, 386–396 (1944); these Rev. **6**, 5] on the evaluation of various integrals on the Wiener measure space C of continuous functions $x(t)$ (Brownian movement measure). The integral over this space of

$$F[x(t)] \cdot \exp\left(\lambda \int_0^1 p(s) x(s)^2 ds\right)$$

is evaluated, where F is a function defined on C, and $p(t)$ is a continuous positive function of t. The evaluation is made in terms of the solutions of the differential equation

$$f''(t) + \lambda p(t) f(t) = 0.$$

Various interesting special cases are examined.

J. L. Doob (Washington, D. C.).

***Palm, Conny. Intensitätsschwankungen im Fernsprechverkehr.** Ericsson Technics no. **44**, 189 pp. (1943). [MF 11044]

A systematic analysis of special stochastic processes related to various technical problems of telephone traffic. Special attention is paid to variations of intensity, various after-effects, the compound Poisson process, etc. In the last part special methods are developed for a comparison of the theory with actual measurements. Tables and diagrams illustrate such a comparison with recent Swedish experience. The technical nature of the problems, the new classifications and the multitude of new notions render a short résumé of the results impossible. The table of contents follows.

I. Der Fernsprechverkehr als stochastischer Prozess. (Allgemeine Eigenschaften der Punktprozesse; Sperrung beim Verkehr mit begrenzter Nachwirkung; allgemeine Verfahren zur Berechnung der Sperrung.) II. Intensitätsschwankungen als Ausgangspunkt für Methoden zur Behandlung von Fernsprechverkehrsproblemen. (Träge Intensitätsschwankungen; Eigenschaften summierter Verkehrsfunktionen; schnelle Intensitätsschwankungen; Überlagerung; Eigenschaften bei Verkehren in den Normalformen; Grundsätzliches über die Berechnung von Intensitätsschwankungen aus Messergebnissen.) III. Bericht über ausgeführte Messungen. (Messverfahren und Messvorrichtungen; Methoden zur Behandlung des Messmaterials; Messgenauigkeit; Messergebnisse und Schlussfolgerungen.) IV. Tabellen und Diagramme. *W. Feller.*

Levert, C., and Scheen, W. L. Probability fluctuations of discharges in a Geiger-Müller counter produced by cosmic radiation. Physica **10**, 225–238 (1943). [MF 10732]

The authors consider a stochastic process with after-effect which, with slight modifications, appears in several applications. Suppose that random events occur according to the Poisson law; they are registered by a counter if, at the moment of occurrence of the event, the counter is free.

Now each registration takes a certain constant time τ, the resolving time, during which the counter is "busy." If an event occurs during that period, it is not registered but nevertheless keeps the counter busy during an interval τ (which will overlap the preceding one). The counter can, therefore, theoretically remain busy for an arbitrarily long period. The authors study, by combinatorial methods, the probability distribution of the number of actually registered events. In particular, they find that the variance may differ essentially from the expectation. (This observation is important since it implies that cosmic rays may be "random events" in the sense of the Poisson law even though the registrations seem to contradict the theory.) *W. Feller.*

Kosten, L. On the frequency distribution of the number of discharges counted by a Geiger-Müller counter in a constant interval. Physica **10**, 749–756 (1943). [MF 11425]

The author shows how the problem treated in the paper reviewed above can be treated by a system of simultaneous differential and integral equations. He obtains, by operational methods, the exact solution while the previous combinational method neglected certain terms (which are practically insignificant). *W. Feller* (Providence, R. I.).

Kronig, R. On time losses in machinery undergoing interruptions. Physica **10**, 215–224 (1943). [MF 10731]

Kronig, R., and Mondria, H. On time losses in machinery undergoing interruptions. II. Physica **10**, 331–336 (1943). [MF 10624]

By means of differential equations a special stochastic process is treated which provides the abstract model of the following situation. There are λ machines, each being subject to random interruptions occurring according to the Poisson law. The time of repair is a random variable with known distribution. There is only one workman repairing the machines, and the repairs take place in the same order as the interruptions. In the first paper averages for the time of repair are computed, while the second paper is concerned with the exact distribution. Emphasis is laid on the finiteness of λ. *W. Feller* (Providence, R. I.).

von Schelling, H. Über die Verteilung der Kopplungswerte in gekreuzten Fernmeldekabeln grosser Länge. Elektr. Nachr. Techn. **20**, 251–259 (1943). [MF 11255]

The usual procedure of recurrent "transpositions" in the pairs of wires of a talking circuit in order to eliminate "cross-talk" between adjacent pairs leads to the following problem in the theory of probability. Let $\{X_\nu\}$ be a sequence of independent nonnegative random variables having the same continuous distribution function $F(x)$. Let

$$S_2=|X_2-X_1|,\quad \cdots,\quad S_{n+1}=|S_n-X_{n+1}|.$$

Required: the distribution function $F_n(x)$ of S_n. It is shown that in some typical cases $F_n(x)$ approaches a limit, and it is conjectured that the same is true in general.

W. Feller (Providence, R. I.).

Koch, A. Das Dualkreuzungsverfahren zur Verminderung der Nebensprechkopplung in Fernmeldekablen. Elektr. Nachr. Techn. **20**, 259–263 (1943). [MF 11256]

Simple elementary probability considerations connected with the problem of the most efficient arrangement of transpositions in talking circuits [cf. the preceding review].

W. Feller (Providence, R. I.).

Mathematical Statistics

Nicholson, C. The probability integral for two variables. Biometrika **33**, 59–72 (1943). [MF 11346]

The volumes under the normalized two-dimensional Gaussian surface over the quadrants $h\leqq x<+\infty$, $k\leqq y<+\infty$ are given in Pearson's "Tables for Statisticians and Biometricians" [London, 1924, Part II, Tables VIII and IX] for all values of h, k from 0.0 to 2.6 at intervals of 0.1 and for all values of the correlation coefficient r from -1 to $+1$ at intervals of 0.05. Those tables of three arguments contain about 20,000 entries. In the present paper a concise table of two arguments containing only 900 entries is presented which, together with a small auxiliary table and the table of normal probability integrals, may be used to replace Pearson's Tables VIII and IX. It should be noted, however, that, in order to obtain from this concise table any volume in Pearson's Tables, at least four entries must be looked up and some simple computations must be performed. The paper contains a derivation of the new table, a description of the methods used in computing it and numerical illustrations of its use. *Z. W. Birnbaum.*

Merrington, Maxine, and Thompson, Catherine M. Tables of percentage points of the inverted beta (F) distribution. Biometrika **33**, 73–88 (1944). [MF 11347]

Let $f_{\nu,\mu}(F)$ denote the probability density of Snedecor's F-distribution with ν and μ degrees of freedom. The tables presented in this paper contain upper percentage points, that is, roots of the equation $J=\int_F^\infty f_{\nu,\mu}(F)dF$, for the percent rates $100J=50$, 25, 10, 5, 2.5, 1, 0.5, and the degrees of freedom $\nu=1, 2, \cdots, 9, 10, 12, 15, 20, 24, 30, 40, 60, 120, \infty$, and $\mu=1, 2, \cdots, 29, 30, 40, 60, 120, \infty$. The results are given to five significant digits.

Z. W. Birnbaum (Seattle, Wash.).

Lehmer, Emma. Inverse tables of probabilities of errors of the second kind. Ann. Math. Statistics **15**, 388–398 (1944). [MF 11757]

Let the variates $y_1, \cdots, y_{f_1}, z_1, \cdots, z_{f_2}$ be normally and independently distributed with variance σ^2. The means of $z_1, \cdots, z_{f_2}$ are all zero, the means $\eta_1, \cdots, \eta_{f_1}$ of $y_1, \cdots, y_{f_1}$ are unknown. The hypothesis $\eta_1=\cdots=\eta_{f_1}=0$ can be tested by the analysis of variance test whose power is a function only of φ, where $\sigma^{-2}\sum\eta_i^2=(f_1+1)\varphi^2$. For a given size of test α the author tabulates $\varphi(f_1, f_2, \alpha, \beta)$, the value of φ for which the ordinate of the power function has the value β. The range of values of the parameters is: $(\alpha, \beta)=(.01, .7), (.01, .8), (.05, .7), (.05, .8)$; $f_1=1(1)10$, 12, 15, 20, 24, 30, 40, 60, 80, 120, ∞; $f_2=2(2)20$, 24, 30, 40, 60, 80, 120, 240, ∞. An analytic discussion of the method of computation is given. The table is of value in designing experiments. *J. Wolfowitz* (New York, N. Y.).

Anderson, T. W., and Girshick, M. A. Some extensions of the Wishart distribution. Ann. Math. Statistics **15**, 345–357 (1944). [MF 11754]

The variates $x_{i\alpha}$ are drawn from N multivariate normal populations ($\alpha=1, 2, \cdots, N$), each of p variates ($i=1, 2, \cdots, p$). The quantities

$$a_{ij}=\sum(x_{i\alpha}-\bar{x}_i)(x_{j\alpha}-\bar{x}_j)$$

represent the sums of squares and products, taken over the N populations. If $E(x_{i\alpha})=\mu_{i\alpha}$ and if t is the rank of the matrix

$$\tau_{ij}=\sum(\mu_{i\alpha}-\bar{\mu}_i)(\mu_{j\alpha}-\bar{\mu}_j),$$

the joint distribution of the a_{ij} was derived by Wishart

[Biometrika 20A, 32–52 (1928)] for the case in which $t=0$. When $t=1$, the authors show that the joint distribution of the a_{ij} is the product of the Wishart distribution and a Bessel function; when $t=2$, the distribution is the Wishart distribution multiplied by an infinite series of Bessel functions. For higher values of t, the distribution is obtained as a multiple integral, for which an explicit form has not yet been found. The distribution of the a_{ij} may also be regarded as a generalization to p variates of the noncentral χ^2 distribution. *W. G. Cochran* (Princeton, N. J.).

Robbins, H. E. On the expected values of two statistics. Ann. Math. Statistics 15, 321–323 (1944). [MF 11327]

Let $Y_1, \cdots, Y_n$ be independent random variables each having $\sigma(y)$ as their distribution function. The author derives the first moments of $B-A$ and $\sigma(B)-\sigma(A)$ when $A=\min(Y_1, \cdots, Y_n)$ and $B=\max(Y_1, \cdots, Y_n)$, using a previously proved general theorem [same Ann. 15, 70–74 (1944); these Rev. 6, 5]. The results are known and can be obtained directly. *M. Kac* (Ithaca, N. Y.).

Dyson, F. J. A note on kurtosis. J. Roy. Statist. Soc. (N.S.) 106, 360–361 (1943). [MF 11334]

A sufficient condition for one frequency curve to have a greater β_2 than another (both having the same mean and standard deviation) is that there are four abscissas $a_1<a_2<a_3<a_4$ such that the difference of the ordinates has alternating signs in the successive intervals $(-\infty, a_1)$, (a_1, a_2), $\cdots$, $(a_4, +\infty)$, and that $\sum a_i$ and $(\mu_{22}-\mu_{11})$ are not of the same sign. *R. L. Anderson* (Princeton, N. J.).

Bosworth, R. C. L. Bessel's formula in relation to the calculation of the probable error from a small number of observations. J. Proc. Roy. Soc. New South Wales 78, 81–83 (1944). [MF 11961]

Bessel's formula for the "probable error of n observations" is

$$s=.6745\{\sum(\bar{x}-x_i)^2/(n-1)\}^{\frac{1}{2}}.$$

It is known to be an underestimate, and Jeffreys [Proc. Roy. Soc. London. Ser. A. 138, 48–55 (1932)] has calculated the precise value. It is shown to be nearly equal to the simpler expression

$$.6745\{\sum(\bar{x}-x_i)^2/(n-1.63)\}^{\frac{1}{2}}.$$

W. Feller (Providence, R. I.).

Irwin, J. O., and Kendall, M. G. Sampling moments of moments for a finite population. Ann. Eugenics 12, 138–142 (1944). [MF 10594]

The authors obtain their moments quite readily and in relatively compact form by (a) using k-statistics, and (b) considering the finite population to be a random sample from an infinite population. This latter device enables them to get the moments directly from the known moments of k-statistics for infinite populations. Results previously obtained by other authors are checked, and some further results are derived among which are the covariances between k_1 and other k-statistics:

$$E(k_r-K_r)(k_1-K_1)=(1/n-1/N)K_{r+1},$$

where n is the sample size, N is the population size and K_r is the rth cumulant of the population.

Readers interested in this problem will wish to refer to an entirely different but quite effective device for simplifying the derivation of these moments given by Cornfield [J. Amer. Statist. Assoc. 39, 236–240 (1944); these Rev. 6, 91]. *A. M. Mood* (Princeton, N. J.).

Hoel, Paul G. The accuracy of sampling methods in ecology. Ann. Math. Statistics 14, 289–300 (1943). [MF 9146]

Expressions are given for the variance of estimates of total plant coverage obtained by several sampling methods usual in ecology. The expressions are derived on the assumptions: (1) that plants possess circular crowns and (2) that plants are randomly and independently distributed. The types of sampling unit considered are (1) the quadrat, in which small square areas are examined, and (2) the transect, in which straight lines are laid down through the area and examined. In either case, coverage (the area covered, or the length of line covered) or abundance (the number of plants in the sample) may be recorded. From the practical point of view, it seems likely that the assumption of random distribution will seldom be satisfied.

C. P. Winsor (Princeton, N. J.).

Gumbel, E. J. Ranges and midranges. Ann. Math. Statistics 15, 414–422 (1944). [MF 11759]

Using results from his earlier paper [Ann. Inst. Henri Poincaré 5, 115–158 (1935)] and under the conditions imposed there, the author studies the moment characteristics of the distribution in a sample of n of the mth largest value and of the mth smallest value and, more particularly, their sum and difference, the mth midrange and mth range. It is further assumed that the sample is so large that these two mth extreme values may be regarded as independent. For $m=1$ some of the methods closely parallel those of Fisher and Tippett. For the most part the universe sampled is taken to be symmetric but moment generating functions are given for the asymmetrical case. There is a discussion of the behavior of the distribution of the mth midrange as m increases but there seems to be some obscurity here since under the conditions imposed the mth extremes are virtual extremes, as the author remarks, and n must remain large compared to m.

C. C. Craig (Ann Arbor, Mich.).

Vajda, S. The algebraic analysis of contingency tables. J. Roy. Statist. Soc. (N.S.) 106, 333–342 (1943). [MF 11333]

Aroian, Leo A. Some methods for the evaluation of a sum. J. Amer. Statist. Assoc. 39, 511–515 (1944). [MF 11478]

The author applies elementary sampling methods to the problem of estimating the total of a large set of numbers from the total obtained from a random sample. The improvement of the result by the use of stratified sampling is also discussed. *C. C. Craig* (Ann Arbor, Mich.).

Kempthorne, O. Comments on the note "On a theorem concerning sampling." J. Roy. Statist. Soc. (N.S.) 107, 58 (1944). [MF 11963]

The comments refer to a note by H. Simpson [same J. (N.S.) 106, 266–267 (1943); these Rev. 6, 9]. The author observes that the theorem quoted in the review is not new. He takes exception to some remarks concerning the use of the theorem for statistical tests. *W. Feller.*

Holzinger, Karl J. A simple method of factor analysis. Psychometrika 9, 257–261 (1944). [MF 11549]

In case a correlation matrix can be sectioned into portions of approximate rank unity it is pointed out that the simul-

taneous extraction of correlated factors can be made quite simple. The first centroid coefficients for any one section may be interpreted as structure values; if these are taken for all variables, over all the unit sections, they may be interpreted as structure values for the whole matrix. The work required is illustrated in a numerical example.

C. C. Craig (Ann Arbor, Mich.).

Wilson, E. B. Note on the t-test. Amer. Math. Monthly **51**, 563–566 (1944). [MF 11732]

Wald, A., and Wolfowitz, J. Statistical tests based on permutations of the observations. Ann. Math. Statistics **15**, 358–372 (1944). [MF 11755]

Let $x_1, x_2, \cdots, x_n$ be a sample of n observations. For carrying out tests of hypotheses independent of the distribution of $x_1, \cdots, x_n$, it is necessary to find the distribution of appropriate statistics in a universe of permutations of the observations, where each permutation has the same probability. A sequence $(h_1, \cdots, h_n, \cdots)$ is said to satisfy the condition W if

$$\mu_r(H_n)/[\mu_2(H_n)]^{r/2}=O(1),$$

where $\mu_r(H_n)$ is the rth moment of the numbers $h_1, \cdots, h_n$ around their mean value. Let

$$A_N=(a_1, \cdots, a_N), \quad D_N=(d_1, \cdots, d_N), \quad N=1, 2, \cdots,$$

be sequences satisfying the condition W and let $L_N=\sum_{i=1}^{i=N} d_i x_i$, where the random vector $(x_1, \cdots, x_N)$ takes every value $(a_{i_1}, \cdots, a_{i_N})$ with probability $1/n!$. The authors prove that

$$L_N^*=[L_N-E(L_N)]/\sigma(L_N)$$

is in the limit normally distributed with variance 1. This result remains valid if

$$A_N=(a_{N1}, \cdots, a_{NN}), \quad D_N=(d_{N1}, \cdots, d_{NN}).$$

The authors easily obtain from this result the limit distribution of the rank correlation coefficient [first obtained by H. Hotelling and M. R. Pabst, Ann. Math. Statistics **7**, 29–43 (1936)] and various new results on the limit distribution of certain statistics. Furthermore, they derive under weak restrictions the limit distribution, in the universe of permutations of the observations, of a statistic $W=F(m-1+F)^{-1}$, where F is the analysis of variance ratio [first proposed by B. L. Welch, Biometrika **29**, 21–52 (1937) and E. J. G. Pitman, Biometrika **29**, 322–335 (1938) for the analysis of variance in randomized blocks], and of a statistic T'^2 for testing the hypothesis that two populations have the same mean if the alternatives are restricted to the case that the two populations differ only with respect to their mean values. Moreover, T'^2 is shown to be a monotonic function of Hotelling's T^2.

H. B. Mann.

Kendall, M. G. On autoregressive time series. Biometrika **33**, 105–122 (1944). [MF 10894]

The author discusses the stochastic difference equation

$$u_{t+2}+au_{t+1}+bu_t=\epsilon_{t+2},$$

where ϵ_{t+2} is a chance variable and a and b constants subject to restrictions whose effect is to cause damping. The ϵ's are independent with the same distribution. The solution of this equation is

$$u_t=p^t(A\cos\theta t+B\sin\theta t)+\sum_{j=1}^{\infty}\zeta_j\epsilon_{t-j+1}$$

with

$\zeta_t=(2/(4p^2-a^2)^{\frac{1}{2}})p^t\sin\theta t,\ p=+\sqrt{b},\ \theta=\arctan(4b/a^2-1)^{\frac{1}{2}},$

A and B arbitrary constants. It is assumed that the non-random part of the solution has been "damped out of existence." The serial correlation coefficient ρ_k with log k is shown to be

$$\rho_k=p^k\sin(k\theta+\psi)/\sin\psi$$

with $\tan\psi=((1+p^2)/(1-p^2))\tan\theta$.

Several actual economic time series and an artificially constructed one, all of length about 65, which are studied by the author, give observed correlation functions which do not damp as do the population correlation functions. This is explained as due to the shortness of the series. The author discusses "how long a practical series must be for the damping to show itself decisively."

The case where a random element (say an error of observation) is superimposed on the u's is then discussed. The author concludes that when this is so "the fundamental period calculated from the observed regressions will be too short and may be very considerably so." The remainder of the paper is devoted to the problem of estimating a and b and to an appendix on the relationship between variate differences and serial correlations.

J. Wolfowitz.

Bhattacharya, K. N. On a new symmetrical balanced incomplete block design. Bull. Calcutta Math. Soc. **36**, 91–96 (1944). [MF 11840]

A balanced incomplete block design is an arrangement of v objects into b sets of k each such that every object occurs in r of the sets and every pair of objects in λ of the sets. The author gives the first solution D of a design $(v, b, r, k, \lambda)=(25, 25, 9, 9, 3)$. He obtains D from a new solution $\bar{D}$ for $(16, 24, 9, 6, 3)$ by a process of adjunction so that D can be obtained from $\bar{D}$ by the process of block section.

H. B. Mann (Columbus, Ohio).

Bhattacharya, K. N. A new balanced incomplete block design. Science and Culture **9**, 508 (1944). [MF 11902]

The author gives the first solution of an incomplete balanced block design with $v=16$, $b=24$, $r=9$, $k=6$, $\lambda=3$. His solution is remarkable because two of the blocks have four varieties in common; hence it cannot be obtained from a design with $v=b=25$, $r=9$, $k=9$, $\lambda=3$ by the process of block section. It is the first example of a design with this property.

H. B. Mann (Columbus, Ohio).

Ammeter, H. Das Zufallsrisiko bei kleinen Versicherungsbeständen. Mitt. Verein. Schweiz. Versich.-Math. **42**, 155–182 (1942). [MF 11454]

The author considers the cumulative distribution (c.d.) function of the losses due to a group of insurance contracts. This c.d. function is developed into a Bruns series and its coefficients are expressed in terms of the moments. Several numerical examples illustrate the computation of security loadings and security funds.

E. Lukacs (Berea, Ky.).

Goudsmit, S. A., and Furry, W. H. Significant figures of numbers in statistical tables. Nature **154**, 800–801 (1944). [MF 11657]

Furry, W. H., and Hurwitz, Henry. Distribution of numbers and distribution of significant figures. Nature **155**, 52–53 (1945). [MF 11837]

Suppose that the entries in a table correspond to a theoretical frequency (density) distribution $f(x)$. Write

$x=p\cdot 10^n$ with $1\leqq p\leqq 10$. The density distribution of the "significant" number p is

$$(*)\quad F(p)=\sum_{-\infty}^{\infty} f(p\cdot 10^n)\cdot 10^n$$

$$\sim\int_{-\infty}^{\infty} f(p\cdot 10^n)\cdot 10^n dn = p^{-1}\log 10.$$

The fraction of entries with the first significant figure a is

$$\int_a^{a+1} F(p)dp\sim\log_{10}\{(a+1)/a\}.$$

This theory explains a known fact concerning empirical distributions. The second paper studies in detail the goodness of approximation of the integral in (*) to the original sum. *W. Feller* (Providence, R. I.).

TOPOLOGY

Balanzat, Manuel. Compact and separable sets in spaces D_0. Publ. Inst. Mat. Univ. Nac. Litoral 5, 15 pp. (1943). (Spanish) [MF 10727]

The author continues his investigation of "the spaces D_0," which are weakened metric spaces obeying the following axioms for elements x, y, z of D_0: (1) $xy\geqq 0$, (2) $xx=0$, (3) $xy\leqq xz+zy$ [Univ. Nac. Tucumân. Revista A. 2, 169–175 (1941); these Rev. 3, 313]. The present paper is chiefly concerned with a comparison in spaces D_0 of two different definitions of compactness and of separability. It is shown by examples that sets in spaces D_0 may be compact without being perfectly compact, separable without being perfectly separable, and perfectly compact without being closed [for definitions of "perfectly separable" and "perfectly compact" we refer to Fréchet, Les Espaces Abstraits, Gauthier-Villars, Paris, 1928, pp. 190, 195]. The possible asymmetry of the metric allows three possibly different topologies to be defined. In giving an example the author selects one of these. *J. V. Wehausen* (Washington, D. C.).

Shafarevich, I. On the normalizability of topological fields. C. R. (Doklady) Acad. Sci. URSS (N.S.) 40, 133–135 (1943). [MF 11165]

The principal theorem is that the following conditions are necessary and sufficient for the existence of a norm over a topological field: (a) the set R of elements p such that $p^n\rightarrow 0$ for $n\rightarrow\infty$ is open; (b) the set $\bar{R}$ of elements q with $q^{-1}\epsilon R$ is bounded (that is, for any neighborhood U of 0 there is a neighborhood V of 0 with $\bar{R}V\subset U$). If a norm exists it is essentially unique: all possible norms are powers of a given norm. As an application of this theorem it is shown that any locally bicompact field is normalizable. The results of Ostrowski [Acta Math. 41, 271–284 (1917)] are then applied to characterize completely such fields. *J. L. Kelley* (Aberdeen, Md.).

Novák, Jos. Induktion partiell stetiger Funktionen. Math. Ann. 118, 449–461 (1942). [MF 10709]

A system S of functions f defined over an abstract set A determines a neighborhood topology in A in the following way. Given a point $p_0\epsilon A$, a function $f\epsilon S$ and a number $\epsilon>0$, the set of points p satisfying $|f(p)-f(p_0)|<\epsilon$ is called a neighborhood of p_0. It is easy to see that Hausdorff's axioms (A) and (C) are always satisfied and that in addition axiom (B) is satisfied provided the system S is closed with respect to addition of functions.

The author calls this topology the induction of the system S. Furthermore, he considers the limit topology ("L-induction") determined by the system S as follows. The sequence of points $\{p_n\}$ converges to the point q if q is the only point such that $f(q)=\lim f(p_n)$ for all $f\epsilon S$. The relations between the induction and the L-induction for the particular case of the system S consisting of all the functions of two real variables continuous in each variable are discussed. *W. Hurewicz* (Cambridge, Mass.).

Young, Gail S., Jr. On continua whose links are non-intersecting. Bull. Amer. Math. Soc. 50, 920–925 (1944). [MF 11564]

The terminology of this paper is that of R. L. Moore [Foundations of Point Set Theory, Amer. Math. Soc. Colloquium Publ., vol. 13, New York, 1932]. Let M be a compact metric continuum which is not a simple link of itself and no two of whose links intersect. Then (1) the simple links of M form an upper semi-continuous collection G, filling up M, such that G is an acyclic continuous curve every interval of which contains a degenerate element of G; (2) uncountably many of the simple links of M are degenerate, and the set which is the sum of the nondegenerate simple links of M is an F_σ. Conclusion (2) is false if the hypothesis of compactness is replaced by separability even if each link is a compact continuum. For each point p of M we denote by $M(p)$ the set of all points of M which are not separated from p by each of an uncountable set of points. It is shown that every compact metric continuum is either an $M(p)$ or contains uncountably many degenerate $M(p)$ sets of itself.

If M is a locally connected Moore space then (1) for any simple link L of M and any connected domain R intersecting L the set $L\cdot R$ is connected; thus every simple link of M is connected and locally connected; (2) if M is the sum of its simple links, and no two simple links of M intersect, then every improper point of M belongs to a nondegenerate continuum of improper points of M, no two of which lie in the same simple link; (3) if the hypothesis of (2) holds and M is not a simple link of itself, then the simple links of M form an uncountable collection; (4) if M is separable, the nondegenerate simple links of M form a countable collection and each nondegenerate link is separable; (5) the last conclusion in (4) is false without local connectivity; (6) if M is separable and is not a simple link of itself, but is the sum of its simple links, and no two links of M intersect, then uncountably many links of M are degenerate. *D. W. Hall* (College Park, Md.).

Whyburn, G. T. Interior mappings into the circle. Duke Math. J. 11, 431–434 (1944). [MF 11079]

In an earlier paper [same J. 11, 35–42 (1944); these Rev. 5, 213] the author surmised that any continuous transformation f of a cyclic locally connected metric continuum into a circle is homotopic to an interior transformation. The main portion of the present paper is devoted to a proof of this conjecture. An extension to noncyclic locally connected continua is obtained by introducing the monotone retraction r onto the smallest A-set containing all cyclic elements on which f is essential; f is homotopic to a mapping φr such that φ is interior. *R. L. Wilder.*

Gottschalk, W. H. Orbit-closure decompositions and almost periodic properties. Bull. Amer. Math. Soc. 50, 915–919 (1944). [MF 11563]

Let X be a metric space and f a continuous mapping $X \to X$. The author calls f pointwise almost periodic (a.p.) if for every point x and neighborhood $U(x)$ there exists an integer N such that at least one of every N consecutive images of x under powers of f falls in U. Pointwise almost periodicity is uniform if N depends only on the diameter of U and not on x. Theorem: if f is pointwise a.p. (uniformly pointwise a.p.), X possesses a decomposition (continuous decomposition) into sets which are closures of semi-orbits of f; the converse is true if X is locally compact (compact). If f is a homeomorphism, "semi-orbits" can be replaced by "orbits" provided "locally" is omitted. *P. A. Smith.*

Erdös, P., and Stone, A. H. Some remarks on almost periodic transformations. Bull. Amer. Math. Soc. 51, 126–130 (1945). [MF 11827]

The authors generalize slightly and prove in a simple manner certain theorems of Gottschalk about recurrent and almost periodic (here called "strongly almost periodic") homeomorphisms [Bull. Amer. Math. Soc. 50, 222–227 (1944); these Rev. 5, 213]. In addition, they show that a homeomorphism $f\colon X \to X$ of a totally bounded metric space X is strongly almost periodic if its negative powers are equi-uniformly continuous. If both its positive and negative powers are equi-uniformly continuous there exists an increasing sequence of integers $n_1, n_2, \cdots$ such that, over X, $f^{n_i}(x)$ and $f^{m_i}(x)$ $(m_i = n_i^2)$ converge uniformly to x.
P. A. Smith (New York, N. Y.).

Hopf, Heinz. Eine Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze. Portugaliae Math. 4, 129–139 (1944). [MF 11157]

The mapping and covering theorems in question are by (1) Borsuk-Ulam, (2) Alexandroff-Hopf and (3) Lusternik-Schnirelmann-Borsuk. Theorem (2) asserts: if the n-dimensional sphere S^n is covered by $n+2$ closed sets $F_1, F_2, \cdots, F_{n+2}$, of which none contains an antipodal pair of points of S^n, the logical product of any $n+1$ of the F_i is not empty. The author shows that the number π (the angular distance of an antipodal point-pair) has no special force in validating this theorem, so that it remains true when "antipodal pair of points" is replaced by "pair of points at angular distance α," where α is any given number between 0 and π. Furthermore, this theorem generalizes to any closed n-dimensional manifold with a regular Riemannian metric. Theorems (1) and (3) generalize in the same way, but the chief interest in the argument pertains to (2). There is appended a discussion of several related unsolved problems. *H. Blumberg* (Columbus, Ohio).

Chogoshvili, G. The Betti groups of domains of smaller values. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 853–859 (1943). (Georgian. Russian and English summaries) [MF 11710]

In the present note the author considers, under general boundary conditions, the variation of the homology group with integer coefficients of moving domain of smaller values. An English translation will appear in the Travaux de l'Institut Mathématique de Tbilissi, volume 13.
Author's summary.

Heawood, P. J. Note on a correction in a paper on map-congruences. J. London Math. Soc. 19, 18–22 (1944). [MF 11315]

[A note under the same title appeared in the same J. 18, 160–167 (1943); these Rev. 5, 275. The original paper appeared in Proc. London Math. Soc. (2) 40, 189–202 (1935).] The subject of map congruences led the author to the following problem. Consider a sequence of n (not necessarily distinct) symbols, each one of which is either a 1, 2 or 3. Let U_n denote the set of all 3^n such sequences. Let V_n denote any subset of U_n such that every sequence of U_n "avoids" at least one sequence in V_n. When we say that one sequence "avoids" another, we mean that every symbol in one sequence is different from the symbol in the corresponding place of the other sequence. Let $f(n)$ denote the minimum number of sequences which such a V_n must have. The problem is to evaluate $f(n)$. It is easily shown that $f(1)=2$, $f(2)=3$, $f(3)=5$, $f(4)=9$, $f(5)=16$, and that $f(n)<2f(n-1)$. The author gives an upper bound for $f(n)$ somewhat better (and also more complicated) than the obvious $2^{n-1}+1$. It is not known whether the author's upper bound is actually equal to $f(n)$. *D. C. Lewis.*

MATHEMATICAL PHYSICS

Herzberger, M. Studies in optics. II. Analysis of a given system with the help of the characteristic function, using the direct method of analysis. Quart. Appl. Math. 2, 336–341 (1945). [MF 11774]

[The first part appeared in the same Quart. 2, 196–204 (1944); these Rev. 6, 108.] The author introduces Hamilton's characteristic function into his direct method of geometrical optics developed in previous papers [Quart. Appl. Math. 1, 69–77 (1943); Trans. Amer. Math. Soc. 53, 218–229 (1943); these Rev. 4, 204] and with it finds the image of an arbitrary surface in a given optical system with rotational symmetry. As an example, the image formation of a sphere is investigated. *P. Boeder.*

Vekua, Ilja. On the general problem of diffraction. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 4, 503–506 (1943). (Russian. Georgian summary) [MF 11706]

The object of this paper is to point out an error in two papers of V. D. Kupradze [C. R. (Doklady) Acad. Sci. URSS (N.S.) 16, 31–34 (1937); Trudy Tbiliss. Math. Inst. 2, 143–162 (1937)]. The author shows that, if the necessary corrections are made in the integral equations used by Kupradze, these equations no longer can yield a solution of the general problem of diffraction. The problem is thus left wide open. *H. P. Thielman* (Ames, Iowa).

Bethe, H. A. Theory of diffraction by small holes. Phys. Rev. (2) 66, 163–182 (1944). [MF 11309]

The writer treats the diffraction of a harmonic plane electromagnetic wave by a hole (of diameter small compared with the incident wave length) in a plane conducting screen. The discussion is in physical terms and is carried through with great skill. The idea is to find a field consistent with Maxwell's equations and subject to $\overrightarrow{E_T}$ continuous over the hole and 0 on the screen and $\overrightarrow{\Delta H_T}$ equal to the incident $\overrightarrow{H_T}$ at the hole; T denotes the tangential component. It is first shown (as is well known) that the "Kirchhoff type" of solution does not satisfy these conditions. The writer sets

up equations analogous to the usual ones but with electric and magnetic quantities interchanged and then introduces a suitable distribution of fictitious magnetic poles and currents over the surface of the hole. [It may be noted that these ideas go back forty years to Larmor.] The actual determination of the strengths of these entities is feasible with the approximations attendant on the minuteness of the opening. Some interesting qualitative remarks are made about the excitation of cavities coupled by a small hole. It is not altogether clear to the reviewer that, in a rigorous formulation of the method, Maxwell's equations are really satisfied. The difficulty envisaged is connected with the well-known situation for secondary electromagnetic radiation, namely, that when the surface (here the surface over the hole) is not closed then it is necessary in general to add line singularities over the boundary curve (of the hole). *D. G. Bourgin* (Urbana, Ill.).

Geffcken, W. Reflexion elektromagnetischer Wellen an einer inhomogenen Schicht. Ann. Physik (5) **40**, 385–392 (1941). [MF 11193]

A plane wave at normal incidence enters a slab $x_0 \leqq x \leqq x_1$ with refraction index $n(x)$, where x is measured in the propagation direction. Then, with A as the complex amplitude and R_0 as reflection coefficient, one immediately obtains $(d/dx)\log A=((1-R_0)/(1+R_0))-in(x_0)$, $-in(x_1)$ at x_0 and x_1, respectively. This problem has often been treated. The author claims the novelty of expressing the analysis in terms of $(d/dx)\log A=i\phi$. Then the boundary conditions are

$$\phi(x_0)=n(x_0)((1-R_0)/(1+R_0)),\quad \phi(x_1)=n(x_1),$$

while (1) $i\phi'(x)-(n^2-\phi^2)=0$. The variable $Q=C(n-\phi)$ has physical significance. Then (1) is equivalent to an integral equation relating Q and an integral of Q^2. The author discusses some simple forms of $n(x)$, where the Q^2 terms are either neglected or averaged. *D. G. Bourgin.*

Humblet, J. Sur le moment d'impulsion d'une onde électromagnétique. Physica **10**, 585–603 (1943). [MF 11422]

The author shows that up to R^{-2} terms in the fields the moment of momentum flux is approximately (1) $\int\int \vec{r}\times\vec{E}\times\vec{H}dS$ in the usual notation. This can be written in the form (2) $\sum\int\int E_i\vec{r}\times\nabla\vec{A}+\vec{E}\times\vec{A}-\vec{E}\times\nabla(\vec{r}\times\vec{A})dS$, where $\vec{A}$ is the vector potential. This decomposition is claimed as the main novelty of the paper. On taking mean time values the contribution of the last integral to the momentum flux at infinity vanishes. The importance of this is that the R^{-2} terms in the fields enter in this integral only. Hence one needs merely R^{-1} terms in $\vec{A}$ and $\vec{E}$ to compute the mean flux to the approximation given by (1). The two remaining integrals in (2) are analogous to orbital and spin contributions. Consideration is given to dipole radiation and meson field analogies and the familiar difficulties involved in a monochromatic plane wave concept. For the last it is shown that the flux behaves reasonably if one uses a wave packet [as would be expected]. *D. G. Bourgin.*

Feinberg, E. L. On the theory of propagation of radio waves along real surfaces. Bull. Acad. Sci. URSS. Sér. Phys. [Izvestia Akad. Nauk SSSR] **8**, 200–209 (1944). (Russian) [MF 11723]

Chu, En-Lung. Notes on the stability of linear networks. Proc. I. R. E. **32**, 630–637 (1944). [MF 11258]

Nyquist's stability criterion for feedback amplifiers is generalized to the case where the loop gain approaches a finite limit less than 1 as $\omega\to\pm\infty$. The following rule (given without proof by Llewellyn) for bilateral systems is derived. If a two-terminal network is stable when open-circuited (short-circuited) and its $Z(\omega)(A(\omega))$ plot for real ω does not enclose the origin, it will remain stable when short-circuited (open-circuited). Some examples of these results are given and stability in impulsive type networks is considered. *C. E. Shannon* (New York, N. Y.).

de Beauregard, Olivier Costa. Sur l'électromagnétisme des milieux polarisés; définition d'un tenseur de Maxwell asymétrique. C. R. Acad. Sci. Paris **217**, 662–664 (1943). [MF 11677]

Fock, V. A. Electrical field near a depression in a conducting plane. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 343–345 (1943). [MF 11176]

A conducting plane has a depression in the shape of part of a sphere's surface. The sphere's radius is r. An incident electromagnetic wave has wavelength λ, where $r\ll\lambda$. The writer's attack on the problem of the local field pivots on his remark that in a cube of side l, where $r\ll l\ll\lambda$, the field may be considered electrostatic. Since $r\ll l$ the situation in the neighborhood of the depression is as if the boundary conditions were those of the electrostatic field supposed occupying all space. The mathematical solution of the electrostatic problem is carried through with great elegance. *D. G. Bourgin* (Urbana, Ill.).

da Silveira, A. On Helmholtz's electrodynamical potential and the induction coefficients of unclosed currents in quasi-stationary processes. Philos. Mag. (7) **35**, 346–351 (1944). [MF 10887]

The author presents an elaboration of Planck's treatment of electrodynamical potential and induction coefficients of nonclosed currents in quasi-stationary processes, an elaboration in which he desires to avoid certain objections to Planck's treatment occurring in the computation of the magnetic energy of a system of currents

$$W_m^*=(8\pi)^{-1}\int(H,B)dv.$$

The integral is over all space, and H, B are the magnetic field and magnetic induction vectors, respectively. He considers a system of closed or open circuits in vacuo, with variable currents, free of electrical and magnetic polarization, and where the electric field vector E is derivable from a potential function V(=quasi-stationary state). Under these assumptions he derives the expression

$$(1)\quad W_m^*=\tfrac{1}{2}(\mu_0/c^2)\int\int|J|\,|J'|((\cos\omega)/r+\tfrac{1}{2}\partial^2 r/\partial s\partial s')dvdv',$$

where r is the distance and ω the angle between the (distinct) linear elements of current ∂s, $\partial s'$ in dv, dv'; J, J' are current densities at ∂s, $\partial s'$; μ_0 is the magnetic permeability. Several specializations of (1) are indicated, principally to open and to almost uniform closed currents.

A. L. Foster (Berkeley, Calif.).

Markov, M. A. On the relativistically invariant "cutting-off factors" in electrodynamics. C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 18–19 (1943). [MF 11185]

With the help of the Dirac-Fock-Podolsky form of quantum-electrodynamics (in which different time variables for the radiation field and for material particles are used) some incompatibilities are pointed out if relativistically invariant "cutting-off factors" are introduced in the Fourier series of the vector potentials. *W. Pauli.*

Born, Max, and Peng, H. W. Quantum mechanics of fields. II. Statistics of pure fields. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 92–102 (1944). [MF 11160]

[The first part appeared in the same Proc. **62**, 40–57 (1944); these Rev. **5**, 224; cf. the following review.] A new kind of statistics is proposed for a system of oscillators which describe a pure field (for instance, the electromagnetic radiation field without interaction with charged particles). The admitted degrees of freedom, especially the admitted wave-vectors k in a large hole of volume Ω, are not considered as known a priori, but an a priori possibility is introduced for every k fulfilling the boundary conditions to be admitted ($\Delta_k=1$) or not ($\Delta_k=0$). For the field oscillators with an admitted k the term "apeiron" is suggested. It has as many degrees of freedom as the number of independent polarizations for a real field and twice this number for a complex field. It is pointed out that, instead of considering the field variables as q-numbers depending on the c-number parameters k_x, k_y, k_z, it is also possible to consider k_x, k_y, k_z as q-numbers which commute with each other and with all field variables, and to treat the latter quantities as diagonal matrices with respect to the eigenvalues of the k. This possibility (which uses reducible representations for the observables) exists both in the usual form of the theory and in the new form here proposed.

In order to make the total zero point energy finite, distributions of apeirons (admitted k values of oscillators) with a given finite total number $n=\sum_k\Delta_k$ of apeirons and a given total zero point energy $U'=\sum_k\Delta_k\sum_r\epsilon_k$ are considered. Here $\sum_r$ means summation over all polarizations including an additional factor $\frac{1}{2}$ for real fields, and ϵ_k the energy hN_k of one quantum with a wave vector k.

The canonical distribution of apeirons and quanta is discussed in detail. In the general case of a complex field this distribution is obtained by means of the grand partition function

$$\Xi=\sum \exp \Big[-\beta \sum_k \sum_r \Delta_k(N_{kr+}+N_{kr-})\epsilon_k \\ -\gamma \sum_k \sum_r \Delta_k(N_{kr+}-N_{kr-})-\beta' \sum_k \Delta_k\bar{\omega}\epsilon_k+\zeta \sum_k \Delta_k\Big],$$

where the first sum is extended over all quantum numbers $\Delta_k=0$ or 1, and $N_{kr\pm}=0, 1, 2, \cdots$ and $\bar{\omega}$ is the number of independent polarizations. Different assumptions are possible for β', the value of which is not necessarily connected with β, which is inversely proportional to the ordinary temperature. *W. Pauli* (Princeton, N. J.).

Born, Max, and Peng, H. W. Quantum mechanics of fields. III. Electromagnetic field and electron field in interaction. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 127–137 (1944). [MF 11546]

[Cf. the preceding review.] In sections 1–4 the usual quantization of Maxwell's field and Dirac's field in interaction is developed in a form where space and time are eliminated. This is possible by giving the total energy (Hamiltonian) and the total momentum of the fields in a given volume Ω as functions of the coefficients of their Fourier expansions with respect to the space coordinates, and by giving also the commutation rules between these Fourier coefficients. For all problems which concern only the volume Ω as a whole and stationary states, it is, however, not necessary to add the assumption that the quantities in question are Fourier coefficients, but only that they satisfy certain commutation laws and that the total energy and the total momentum are known functions of them. In section 5 the remark is made that the wave vectors k and l of the electromagnetic field and the material field can also be considered as operators commuting with all field quantities, and summations over k and l can be replaced by traces over (reducible) submatrices with respect to the eigenvalues of k and l.

A connection with the statistics developed in part II is not yet given nor are new results concerning self energies (divergence difficulties of the usual theories) derived. *W. Pauli* (Princeton, N. J.).

Belinfante, F. J. On the commutativity of the Dirac electron wave-function with the electromagnetical field. Physica **10**, 720–724 (1943). [MF 11424]

Nielsen, Harald H. The quantum mechanical Hamiltonian for the linear polyatomic molecule treated as a limiting case of the non-linear polyatomic molecule. Phys. Rev. (2) **66**, 282–287 (1944). [MF 11419]

Madhava Rao, B. S. Symposium on modern algebra and theory of elementary particles. Math. Student **12**, 30–58 (1944). [MF 11850]

Petiau, Gérard. Sur la représentation d'interactions s'exerçant par l'intérmediaire de la particule de spin total maximum $2.h/2\pi$. C. R. Acad. Sci. Paris **217**, 665–667 (1943). [MF 11678]

Hulthén, Lamek. On the meson field theory of nuclear forces and the scattering of fast neutrons by protons. Ark. Mat. Astr. Fys. **29A**, no. 33, 22 pp. (1943). [MF 11541]

Hulthén, Lamek. On the meson field theory of nuclear forces and the scattering of fast neutrons by protons. II. Ark. Mat. Astr. Fys. **30A**, no. 9, 16 pp. (1943). [MF 11542]

Hulthén, Lamek. On the meson field theory of nuclear forces and the scattering of fast neutrons by protons. III. Ark. Mat. Astr. Fys. **31A**, no. 15, 11 pp. (1944). [MF 11784]

Hulthén, Lamek. Nuclear forces in a non-symmetrical scalar-pseudoscalar meson field theory. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 14, 4 pp. (1944). [MF 11539]

Hulthén, Lamek. On the virtual state of the deuteron. Ark. Mat. Astr. Fys. **29B**, no. 1, 11 pp. (1942). [MF 11543]

Holmberg, Bengt. On nuclear forces described by a pseudovector-scalar neutral meson field. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **14**, no. 22, 6 pp. (1944). [MF 11538]

Klein, O. On the meson pair theory of nuclear interaction. Ark. Mat. Astr. Fys. **30A**, no. 3, 13 pp. (1944). [MF 12003]

Jauch, J. M. Neutron-proton scattering and the meson theory of nuclear forces. Phys. Rev. (2) 67, 125–132 (1945). [MF 12214]

Alfvén, Hannes. On the motion of a charged particle in a magnetic field. Ark. Mat. Astr. Fys. **27A**, no. 22, 20 pp. (1941). [MF 12025]

Schönberg, Mario. Theory of the point electron. Anais Acad. Brasil. Ci. **16**, 323–326 (1944). (Portuguese) [MF 12213]

Statistical Mechanics

Thiruvenkatachar, V. R. Connection between spin and statistics. Math. Student **12**, 59–62 (1944). [MF 11851]

Wang, J. S. Approximate partition function in generalized Bethe's theory of superlattices. Phys. Rev. (2) 67, 98–106 (1945). [MF 11965]

Krylov, N. Relaxation processes in statistical systems. Nature **153**, 709–710 (1944). [MF 10884]

The author contemplates the problem of the "foundation of statistics" (regarded as "the establishment of the connection between statistics and mechanics"), and, in particular, the notion of the relaxation process (that is, the passage from a given statistical state to one of equilibrium). He sets aside the classical discussions of the subject as either being full of contradictions or as not solving the problem in question. In particular, he writes: "In the present investigation, the notion of ergodicity is ignored. I reject the ergodic hypothesis completely: it is both insufficient and unnecessary for statistics. I use, as starting point, the notion of motions of the mixing type, and show that the essential mechanical condition for the applicability of statistics consists in the requirement that in the phase space of the system all the regions with a sufficiently large size should vary in the course of time in such a way that while their volume remains constant, according to Liouville's theorem, their parts should be distributed over the whole phase space (more exactly, over the layer corresponding to given values of the single-valued integrals of the motion) with a steadily increasing degree of uniformity."

It is stated in qualitative outline that all the essential occurrences of statistical principles in statistical mechanics derive from the operation of mixture (for example, the H-theorem) and the relaxation process is visualized as the process of mixing of the initial region, corresponding to the original nonequilibrium state, over the whole surface of given values of the single-valued integrals of the motion. The relaxation time t is defined as the time of this mixing, that is, until it reaches such a degree of uniformity as corresponds to the degree of accuracy of macroscopic measurement. The author gives (without proof) the expression for t in the case of an ideal gas:

$$t=\frac{3\tau/2}{\ln \lambda/r_0}\left\{\ln\frac{2\pi}{(\Delta p/p_0)}\right\},$$

where τ and λ are the duration and length of the mean free path, r_0 is the radius of a molecule, $p_0=(mkT)^{\frac{1}{2}}$, $\Delta p=A/L$, L is the linear dimension of the system, and $A\sim h$; it thus proves to be of the order of a few τ. In a transition to the classical case, $A\to 0$, $t\to\infty$.

The whole paper is written in the form of the outline of a program. It is the reviewer's opinion that the real difficulties in statistical theories are encountered not in the assembling of such qualitative suggestive pictures, but at the stage when they are rendered as absolutely precise mathematical statements to be proved rigorously. He also finds it difficult to understand the author's complete rejection of the notion of ergodicity, in view of the close connection between it and the property of being of the mixing type. Thus the mixture property implies ergodicity (but not conversely) so any difficulty in establishing the latter in a given system must a fortiori occur in establishing the former. Moreover, the mixture property is equivalent to ergodicity in the symmetric product of the phase-space by itself. *B. O. Koopman* (Washington, D. C.).

Wataghin, G. Statistical mechanics at extremely high temperatures. Phys. Rev. (2) 66, 149–154 (1944). [MF 11155]

This paper discusses certain phenomena of statistical mechanics occurring at extremely high temperatures. The range of temperatures below the limit $T_u\sim k^{-1}\,137mc^2$ is divided into three intervals. We have in all: (1) temperatures well below the critical temperature $T_0=k^{-1}mc^2$ (for example, $T<k^{-1}mc^2/10$) for which the usual quantum statistical laws hold; (2) temperatures T: $\frac{1}{4}k^{-1}mc^2\leqq T\leqq T_0$, characterized by the appearance of positrons and electrons produced by thermal photons, at which an approximate treatment of the assembly by means of elementary statistical formulae is still possible; (3) temperatures T: $T_0<T<137T_0$ which can be introduced only for assemblages of heavy particles, such as nuclei; (4) conditions $T\gg k^{-1}10^8$ev at which the average amount of energy per particle is so great that the majority of the processes become irreversible and no equilibrium is possible. The lack of equilibrium is due to the simultaneous or successive creation of many particles (especially of unstable mesotrons) and to the processes involving the emission of neutrinos, in which an appreciable fraction of the energy and momentum escapes all observation.

It is found that, because of the behavior of some high energy particles, known experimentally from observations on cosmic rays and nuclei, some limitations arise to the validity of the laws of quantum statistics, in accordance with the idea of the existence of a supplementary indeterminacy for high energy particles and a lower limit for measurable lengths. All this is taken into account by tentative assumptions in developing by conventional combinatorial methods some simple formulae covering the different cases.

Some astronomical aspects of the phenomena of pair production and of the gravitational effect of light particles are discussed. Speculations concerning the pre-stellar state of matter and the expanding universe are made.

B. O. Koopman (Washington, D. C.)

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AUTHOR INDEX

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